Problem of the Week 10, Fall 2008

The *void cube* is a solid obtained by drilling square-shaped holes, all the way through, in each of the faces of a cube. (Figure 1.) It is known that the surface of the void cube can be bijectively and continuously deformed to an n-fold torus. Figure 2 shows a 2-fold torus and a 4-fold torus. What is n? Justify your answer.

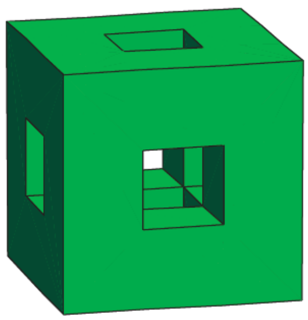

Figure 1

Figure 2

Solution by organizers. (Combinatorial). In every outer face of the void cube, add a segment from each vertex to its closest vertex in the inner square. This creates a partition of the surface of the void cube into quadrilaterals. There are 4 quadrilaterals in each of the six outer faces of the void cube and four faces surrounding each of the six holes. This gives a total of 48 faces. Because each face has exactly four edges, and each edge is in exactly two faces, there are exactly $48 \cdot 4/2 = 96$ edges. Finally, there are 8 outer vertices (the vertices of the outer cube), 4 vertices in each of the six outer faces of the cube, for a total of 24, and 8 vertices located inside the outer cube (they in fact form a little cube). This gives 40 vertices. Then we have a polygonization of the surface of the void cube, consisting of 40 vertices, 96 edges, and 48 faces. According to the Euler-Poincare characteristic, if the genus of the surface is equal to g, then

$$\begin{aligned} V - E + F &= 2 - 2g \\ 40 - 96 + 48 &= 2 - 2g \\ g &= 5. \end{aligned}$$