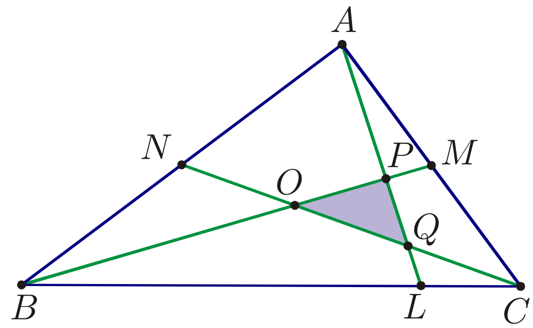
Problem of the Week 9, Fall 2008

Triangle ABC has $AC = 9, AB = 12$, and $BC = 15$. The points M and N are the midpoints of the segments AC and AB, respectively. A point L is constructed on segment BC, such that $LC = 3$.

Segments BM and CN intersect at O, and segment AL intersects BM and CN at P and Q, respectively. What is the area of triangle OPQ?



Solution by organizers. Because $9^2 + 12^2 = 15^2$, then ABC is a right triangle. We use coordinate geometry placing A on the origin, B on the positive x-axis, and C on the positive y-axis. Then, $A = (0, 0)\,, B = (12, 0)\,, C = (0, 9)\,, M = \left(0, \frac{9}{2}\right), N = (6, 0)$. To find the coordinates (L_x, L_y) of L, let R be the point on AC such that $L'L$ is horizontal. Triangles ABC and RLC are similar and thus $\frac{RL}{AB} = \frac{LC}{BC} = \frac{CR}{CA}$. That is, $\frac{L_x}{12} = \frac{3}{15} = \frac{9-L_y}{9}$. This gives $L = (L_x, L_y) = \left(\frac{12}{5}, \frac{36}{5}\right)$. To find the coordinates of O, P, and Q, we find the equations of the lines AL, BM, and CN.

$$\begin{aligned}
\overleftrightarrow{AL} &: \; y = 3x, \\
\overleftrightarrow{BM} &: \; y = -\frac{3}{8}x + \frac{9}{2}, \\
\overleftrightarrow{CN} &: \; y = -\frac{3}{2}x + 9.
\end{aligned}$$

The three intersections are

$$\begin{aligned}
O &= \overleftrightarrow{BM} \cap \overleftrightarrow{CN} = (4, 3)\,, \\
P &= \overleftrightarrow{AL} \cap \overleftrightarrow{BM} = \left(\frac{4}{3}, 4\right), \\
Q &= \overleftrightarrow{AL} \cap \overleftrightarrow{CN} = (2, 6)\,.
\end{aligned}$$

Finally, the area of triangle PQR is

$$\frac{1}{2}\left|\det\begin{pmatrix} 4 & \frac{4}{3} & 2 \\ 3 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}\right| = 3.$$