Problem of the Week 7, Fall 2008

Let $n > 0$ be a natural number. Determine all n-degree polynomials $P(x)$ with n positive real roots such that

$$P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + 1$$

and $a_1 = a_{n-1} = n - 10$.

Solution by organizers. We show that there are two possibilities: either $P(x) = x^2 - 8x + 1$ or $P(x) = x^4 - 6x^3 + a_2x^2 - 6x + 1$ with $10 < a_2 < 11$.

Suppose that $r_1, r_2, ..., r_n$ are the n positive real roots. Then $P(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$, which implies

$$r_1 + r_2 + \ldots + r_n = -a_{n-1} = 10 - n \qquad \text{and} \qquad r_1r_2\ldots r_n = (-1)^n = 1.$$

Since the arithmetic mean of a set of positive reals is always at least as large as its geometric mean, we must have

$$\frac{10 - n}{n} = \frac{r_1 + r_2 + \ldots + r_n}{n} \geq \sqrt[n]{r_1r_2\ldots r_n} = 1,$$

that is, $n \leq 5$. By Decarte's rule of signs the coefficients of an n-degree polynomial with n positive real roots must alternate signs. Because the leading and constant coefficients are both equal to $1 > 0$, then n must be even. These two observations leave 2 and 4 as the only possible values of n.

For $n = 2$, we have $a_{n-1} = a_1 = n - 10 = -8$. Thus $P(x) = x^2 - 8x + 1$, which in fact has 2 positive real roots: $8 + \sqrt{15}$ and $8 - \sqrt{15}$.

For $n = 4$, we have $a_{n-1} = a_1 = n - 10 = -6$. Thus $P(x) = x^4 - 6x^3 + a_2x^2 - 6x + 1$. Divide by x^2 and write $y = x + \frac{1}{x}$. Note that $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$.

$$\begin{aligned} \frac{P(x)}{x^2} &= x^2 + \frac{1}{x^2} - 6\left(x + \frac{1}{x}\right) + a_2 \\ &= \left(x + \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + a_2 - 2 \\ &= y^2 - 6y + a_2 - 2. \end{aligned}$$

The roots of P are the solutions $\frac{1}{2}\left(r \pm \sqrt{r^2 - 4}\right)$ to the equation $r = x + \frac{1}{x}$, where $r = 3 \pm \sqrt{11 - a_2}$ is a root of $y^2 - 6y + a_2 - 2$. Note that $\frac{1}{2}\left(r \pm \sqrt{r^2 - 4}\right) > 0$ as long as $r > 0$ and $r^2 > 4$. This means $a_2 < 11$ and $3 + \sqrt{11 - a_2} > 3 - \sqrt{11 - a_2} > 2$, that is $a_2 > 10$.