

Problem of the Week 6, Fall 2008

There is a fortune-teller at the annual village fair. Nine fortune-tellers out of ten are always right; the other ones are always wrong. Nobody at the fair knows which sort of fortune-teller is there this year. Another annual feature of the fair is the lottery. Each of the twenty boys in the village has an equal chance to be chosen as the Harvest King; and each of the twenty girls has an equal chance to be chosen as the Harvest Queen.

(a) Henry asked the fortune-teller whether he would be chosen, and was told that he would. What are his chances now?

(b) As he came out, Henry met his friend Anne. He told her about the prediction. She immediately went into the tent and asked whether she would be chosen as Harvest Queen. The fortune-teller told her that she would be chosen. What are her chances now?

Solution by organizers. (a) There are two ways for Henry to be told that he will be king: either the fortune teller is always right and he in fact will be king (this happens $\frac{9}{10} \left(\frac{1}{20}\right)$ of the time), or the fortune teller is always wrong and he will not be king (this happens $\frac{1}{10} \left(\frac{19}{20}\right)$ of the time). Thus the probability that Henry will be king given that he was told so by the fortune teller is

$$\frac{\frac{9}{10} \left(\frac{1}{20}\right)}{\frac{9}{10} \left(\frac{1}{20}\right) + \frac{1}{10} \left(\frac{19}{20}\right)} = \frac{9}{28}.$$

(b) Again, there are two ways for Henry and Anne to be told that they will be king and queen by the same fortune teller: either the fortune teller is always right and they in fact will be king and queen (this happens $\frac{9}{10} \left(\frac{1}{20}\right) \left(\frac{1}{20}\right)$ of the time), or the fortune teller is always wrong and he will not be king and she will not be queen (this happens $\frac{1}{10} \left(\frac{19}{20}\right) \left(\frac{19}{20}\right)$ of the time). Thus the probability that Henry will be king and Anne will be queen given that they were both told so by the same fortune teller is

$$\frac{\frac{9}{10} \left(\frac{1}{20}\right) \left(\frac{1}{20}\right)}{\frac{9}{10} \left(\frac{1}{20}\right) \left(\frac{1}{20}\right) + \frac{1}{10} \left(\frac{19}{20}\right) \left(\frac{19}{20}\right)} = \frac{9}{370}.$$

Note. We can rephrase this solution using conditional probabilities. Let A be the event that Henry will be king, B the event that the fortune teller tells Henry that he will be king, C the event that the fortune teller tells Henry that he will be king and Anne that she will be queen.

We are looking for $P(A|B) = \frac{P(A \cap B)}{P(B)}$ in part (a), and $P(A|C) = \frac{P(A \cap C)}{P(C)}$ in part (b). Note that $A \cap B$ will happen only if Henry will be king and the fortune teller is always right, so $P(A \cap B) = \frac{9}{10} \left(\frac{1}{20}\right) = \frac{9}{200}$. On the other hand, the fortune teller tells Henry that he will be king in the two situations listed earlier in part (a), giving $P(B) = \frac{9}{10} \left(\frac{1}{20}\right) + \frac{1}{10} \left(\frac{19}{20}\right) = \frac{28}{200}$. Thus $P(A|B) = \frac{9}{28}$. Similarly, $A \cap C$ happens only if Henry will be king, Anne will be queen, and the fortune teller is always right, so $P(A \cap C) = \frac{9}{10} \left(\frac{1}{20}\right) \left(\frac{1}{20}\right) = \frac{9}{4000}$. On the other hand, the fortune teller tells Henry that he will be king and Anne that she will be queen in the two situations listed earlier in part (b), giving $P(C) = \frac{9}{10} \left(\frac{1}{20}\right) \left(\frac{1}{20}\right) + \frac{1}{10} \left(\frac{19}{20}\right) \left(\frac{19}{20}\right) = \frac{370}{4000}$. Thus $P(A|C) = \frac{9}{370}$.