Problem of the Week 2, Fall 2008

Evaluate

$$\sqrt{\underbrace{111\ldots111}_{2000\text{ digits}} - \underbrace{222\ldots222}_{1000\text{ digits}}}.$$

Your answer should be a number in decimal notation without square roots or any other algebraic operation.

Solution by organizers. Write $x = \underbrace{111\ldots111}_{1000\text{ digits}}$. Then

$$\underbrace{111\ldots111}_{2000\text{ digits}} = \underbrace{111\ldots111}_{1000\text{ digits}}\underbrace{000\ldots000}_{1000\text{ digits}} + \underbrace{111\ldots111}_{1000\text{ digits}} = 10^{1000}x + x$$

and

$$\underbrace{222\ldots222}_{1000\text{ digits}} = 2x.$$

We want the value of

$$\begin{aligned}\sqrt{10^{1000}x + x - 2x} &= \sqrt{(10^{1000} - 1)\,x} = \sqrt{(\underbrace{999\ldots999}_{1000\text{ digits}})x} \\ &= \sqrt{(9x)\,x} = \sqrt{9x^2} = 3x = \underbrace{333\ldots333}_{1000\text{ digits}}.\end{aligned}$$

The same works if we replace 1000 by an arbitrary positive integer n. Write $x = \underbrace{111\ldots111}_{n\text{ digits}}$. Then

$$\sqrt{\underbrace{111\ldots111}_{2n\text{ digits}} - \underbrace{222\ldots222}_{n\text{ digits}}} = \sqrt{10^n x + x - 2x} = \sqrt{(\underbrace{999\ldots999}_{n\text{ digits}})x} = \sqrt{9x^2} = 3x = \underbrace{333\ldots333}_{n\text{ digits}}.$$