Problem of the Week 1, Fall 2008

Find an integer number such that the number formed by its first three digits is a multiple of 3; the number formed by its second, third, and fourth digits is a multiple of 4; the number formed by its third, fourth, and fifth digits is a multiple of 5; and so on. Moreover, no three consecutive digits are zeros.

For example, 288402 is a solution because 288, 884, 840, and 402 are multiples of 3,4,5, and 6 respectively.

Solution by organizers. The largest solution is the 18-digit number

$$999,600,729,024,765,612.$$

In fact, as we show below, there are 300 solutions with 18 digits each.

Assume that $d_1, d_2, d_3, \ldots, d_i, \ldots$is a solution. Because $d_8d_9d_{10}$ is divisible by 10, then $d_{10} = 0$. Also, since $d_{10}d_{11}d_{12} = 0d_{11}d_{12}$ is a multiple of 12, then $d_{11}d_{12}$ is a multiple of 12. Since there are no three consecutive zeros, $d_{11}d_{12} \in \{12, 24, 36, 48, 60, 72, 84, 96\}$. We extend each of these values to $d_{13}, d_{14}, \ldots$, one digit at a time. Note that there is always at most one possibility to add one more digit. This is true because for any integer $k \geq 10$, any two consecutive multiples of k are at distance exactly k which is at least 10. Thus two of them cannot fit on the interval $[ab0, ab9]$ of length 9. (All 3-digit integers starting with ab, namely $ab0, ab1, ab2, \ldots, ab9$; are on this interval.) Our only possibilities are

d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}
0	1	2						
0	2	4	7	6	5	6	1	2
0	3	6	4	4				
0	4	8	1	2	0	8	5	
0	6	0						
0	7	2	8	0				
0	8	4	5					
0	9	6	2					

These possibilities can no longer be extended. For example, $12d_{19}$ cannot be a multiple of 19 because 114 and 133 are consecutive multiples of 19; and $62d_{14}$ cannot be a multiple of 14 because 616 and 630 are consecutive multiples of 14. This proves that any solution has at most 18 digits. Moreover, any solution with 18 digits must satisfy $d_{10}d_{11} \ldots d_{18} = 024765612$.

Now we extend $d_{10}...d_{18} = 024765612$ working back from d_{10} to d_1. For $d_9d_{10}d_{11} = d_902$ to be a multiple of 11, we need d_9+2 to be a multiple of 11. Thus $d_9 = 9$. Now note that d_4, d_6, and d_8 must be even because $d_2d_3d_4, d_4d_5d_6$, and $d_6d_7d_8$ are multiples of $4, 6$, and 8, respectively. Also, $d_5 = 0$ or 5 because $d_3d_4d_5$ is divisible by 5. Now, since 9 divides d_7d_89, then $d_7 + d_8 + 9$ is a multiple of 9.

That is, $d_7 + d_8 = 0, 9$, or 18 with d_8 even. So $d_7d_8 \in \{00, 18, 36, 54, 72, 90\}$. Moreover, since $d_6d_7d_8$ is a multiple of 8, and thus a multiple of 4, then d_7d_8 is a multiple of 4; leaving $d_7d_8 \in \{00, 36, 72\}$. But $d_6 36$ is not a multiple of 8 if d_6 is even, so $d_7d_8 = 00$ or 72. Now, $d_5d_6d_7 = 10\,(d_5d_6) + d_7$ is a multiple of 7. Since $d_7 = 0$ or 7 is a multiple of 7; and 7 does not have factors in common with 10, then d_5d_6 is a multiple of 7. The only 2-digit multiples of 7 starting with $d_5 = 0$ or 5 are 00, 07, and 56. Since d_6 is even and we cannot have three consecutive zeros, the only possible values of $d_5d_6d_7d_8$ are 5600, 0072, and 5672. To determine d_4, note that $d_4 + d_5 + d_6$ is a multiple of 3 because $d_4d_5d_6$ divisible by 6 (and thus by 3). We know d_4 is even and $d_5d_6 = 00$ or 56, thus $d_4d_5d_6 = 456$ or 600. Now, d_3d_4 must be a multiple of 4 because $d_2d_3d_4$ is a multiple of 4. And $d_3 4$ is divisible by 4 if and only if d_3 is odd; and $d_3 6$ is divisible by 4 if and only if d_3 is even. There are 300 multiples of 3 with three digits, namely all integers between $102 = 3\,(34)$ and $999 = 3\,(333)$. Half, 150 of them, are odd. The other half are even. The table below summarizes this analysis.

	$d_4 + d_5 + d_6$ is a multiple of 3			
$d_1d_2d_3$ multiple of 3	d_4 even	$d_5 = 0$ or 5	d_6 even	$d_7d_8 = 00$ or 72
even	4	5	6	00
odd	6	0	0	72
even	4	5	6	72

Each line in the previous table is an 18-digit solution to our problem, and there are 150 of them per line. This gives a total of 450 solutions with eighteen digits. The largest must start with 999, the largest 3-digit multiple of 3. This gives the largest solution 999, 600, 729, 024, 765, 612.