# Problem of the Week 5, Fall 2006

**Solution by organizers.** Suppose that the points $(n, m)$ and $(p, q)$, with $n, m, p$, and $q$ integers, are on a circle with center $\left(\sqrt{2}, \sqrt{3}\right)$. Then $\left(\sqrt{2}, \sqrt{3}\right)$ is on the perpendicular bisector of the segment $(n, m)\,(p, q)$. This perpendicular bisector passes through the midpoint $\left(\frac{n+p}{2}, \frac{m+q}{2}\right)$ and has slope $\frac{-1}{\left(\frac{m-q}{n-p}\right)} = \frac{p-n}{m-q}$. Then the equation of the perpendicular bisector is

$$y = \frac{p-n}{m-q}\left(x - \frac{n+p}{2}\right) + \frac{m+q}{2}$$

which can be written as

$$2\,(m-q)\,y = 2x\,(p-n) + \left(n^2 - p^2 + m^2 - q^2\right).$$

Since the point $\left(\sqrt{2}, \sqrt{3}\right)$ is on this line then

$$2\,(m-q)\,\sqrt{3} = 2\sqrt{2}\,(p-n) + \left(n^2 - p^2 + m^2 - q^2\right).$$

Squaring both sides of this equation and isolating $\sqrt{2}$ we get

$$\begin{aligned} 12\,(m-q)^2 &= 8\,(p-n)^2 + \left(n^2 - p^2 + m^2 - q^2\right)^2 + 4\sqrt{2}\,(p-n)\left(n^2 - p^2 + m^2 - q^2\right) \\ \sqrt{2} &= \frac{12\,(m-q)^2 - 8\,(p-n)^2 - \left(n^2 - p^2 + m^2 - q^2\right)^2}{4\,(p-n)\left(n^2 - p^2 + m^2 - q^2\right)}. \end{aligned}$$

Since $n, m, p$, and $q$ are all integers then the right hand side of this identity is a rational number. But we know $\sqrt{2}$ is irrational, so we have a contradiction.