Problem of the Week 2, Fall 2006

Solution by organizers. We prove that the first $n-1$ terms of A_n are equal to n if and only if n is a prime.

Let $A_n(k)$ denote the k^{th} term of A_n and consider the sequence $A_1(k), A_2(k), A_3(k), ..., A_n(k)$. According to the rules

$$A_n(k) = \begin{cases} A_{n-1}(k) + 1 \text{ if } n \text{ divides } k \\ A_{n-1}(k) \text{ otherwise,} \end{cases}$$

this sequence is nondecreasing and any increment is exactly by 1.

If the first $n-1$ terms of A_n are equal to n then $A_n(n-1) = n$. Since $A_1(n-1) = n-1$ and $A_2(n-1) = n$ then $A_2(n-1) = A_3(n-1) = ... = A_n(n-1) = n$. This means that none of the numbers in $\{2, 3, ..., n-1\}$ divide n. Thus n is prime.

Now assume n is a prime. Since $A_2(n-1) = n$ and the only divisor of n greater than 1 is n then $A_n(n-1) = n$.

Now we prove that the sequence A_n is nondecreasing. Suppose by contradiction that $A_n(i) > A_n(j)$ for some $i < j$. Look at the sequences

$$A_1(i), A_2(i), A_3(i), ..., A_n(i) \text{ and } A_1(j), A_2(j), A_3(j), ..., A_n(j).$$

Both are nondecreasing with differences between consecutive terms of 0 or 1. Since $A_1(i) = i < j = A_1(j)$ and $A_n(i) > A_n(j)$ then there is $m < n$ such that $A_m(i) = A_m(j)$. But this means that $A_n(i) = A_n(j)$. (The i^{th} and j^{th} terms must coincide in every sequence after A_m.)

Finally, since $A_n(1) = n$ (in fact $A_i(1) = i$ for all i) and $A_n(n-1) = n$, then $A_n(1) = A_n(2) = ... = A_n(n-1) = n$.