Problem of the Week 13, Fall 2005

Solution by Lucas Lembrick (edited). The answer is 2499.

This problem is basically figuring out how many numbers there are in a set of ranges, and what those ranges are. The first range where the equality in question is true is $[1, 98]$, the second $[101, 197]$, the third $[202, 296]$. The pattern for these ranges is $[101(n-1), 99(n)-1]$ where n is an integer starting with 1 and corresponding to which range it is (the only exception is the first range, for $n = 1$, that starts with 1 because we are looking for positive integers so we should not include 0). At some point these ranges will meet and overlap. When you have a range $[k, l]$ with $k > l$ you can stop counting the amount of numbers in the range because it is zero. Next we need to count how many numbers are in each range. By simply counting the first few you see there are 98 in the first range, 97 in the second 95 in the third and so on. This pattern is easy to find because the amount of elements in a range $[p, q]$ is $q - p + 1$. So in this case the amount of numbers in any of our ranges (except the first one) is

$$99n - 1 - 101(n-1) + 1 = (99 - 101)n + 101 = -2n + 101,$$

which of course is the numbers $98, 97, 95...5, 3, 1$. Now all we have to do is add them together to see how many positive integers the equality in question is true for.

$$\begin{aligned} 98 + 97 + 95 + ... + 5 + 3 + 1 &= (99 + 97 + 95 + ... + 5 + 3 + 1) - 1 \\ &= (99 + 1) + (97 + 3)...(51 + 49) - 1 = 25(100) - 1 = 2499. \end{aligned}$$

To assure yourself that there are 25 of these sums you can always see when $101(n-1) = 99(n) - 1$ and divide that number by 2 because each range is paired with another range. Of course for this equality $n = 50$ and $50/2 = 25$.