Rolling a ball to the nearest player

Solution by Jason Hughes

Problem Fifteen girls are standing on a field and each of them has a ball. All distances between girls are different. Each girl throws her ball to the girl standing closest to her. Show that:

a) one of the girls does not get any ball and

b) every girl gets at most five balls.

Proof (a) (Direct)

Assume that fifteen girls are standing on a field and each of them has a ball. All distances between girls are different. Each girl throws her ball to the girl standing closest to her. We want to prove that one of the girls does not get any ball. Since all the distances between the girls are different there exists a unique smallest distance. Say that the distance between g1 and g2 (two of the girls) is the smallest. In this case, g1 and g2 exchange their balls, that is g1 throws her ball to g2, and g2 throws her ball to g1. Now, consider the remaining 13 girls and the distances between them. Since they are all different, there exists a minimum one. Say that the distance between g3 and g4 is the smallest. Now, g3 throws her ball to g4, g1 or g2, and g4 throws her ball to g3, g2 or g1. Continue this way until all of the balls have been exchanged. At each step the ball can be thrown to one of the previous girls or the other girl in that step. When there is only one girl left (after the 7th step) it's clear that no one will throw any ball to her. Therefore, one girl does not get any ball, which is what we wanted to prove.

Proof (b) (Contradiction)

Assume by contradiction that there exists a girl g* with more than 5 balls thrown, say $n \geq 6$. Number these girls from1 to n so that girl g(i+1) comes after gi clockwise when we put girl g* at the center. Now, d(gi,g(i+1)) > d(gi, g*) since girl gi throws the ball to girl g*, not to g(i+1). Similarly, d(gi,g(i+1)) > d(g(i+1), g*). This shows that in the triangle (gi g* g(i+1)) the angle centered at g* is greater than 60 degrees. This holds for every i, even for i=n by identifying g(n+1) with g1. We know that a complete angle at g* is 360 degrees which is equal to sum of the angles (gi g* g(i+1)) from 1 to n. But each of them is greater than 60, so the sum is greater than 60n. Since we have assumed that $n \geq 6$ then the sum is greater than 360 which gives a contradiction since it should add up to 360.