Problem of the Week 7, Fall 2005

Solution by George Craciun (edited). Let us use a system of rectangular coordinates, and let $A(0,0), B(6,0), C(6,6), D(0,6)$ and $P(x,y)$.The condition $PA^2+PB^2 = 68$ gives

$$x^2 + y^2 + (x-6)^2 + y^2 = 68$$

which is equivalent to

$$(x-3)^2 + y^2 = 5^2. \tag{1}$$

This means that the point P must move on a sector of circle C with radius 5 and center $(3,0)$. The condition $20 \le PC^2 + PD^2 < 44$ gives,

$$20 \le (x-6)^2 + (y-6)^2 + x^2 + (y-6)^2 < 44$$

which is equivalent to

$$10 \le (x-3)^2 + y^2 - 12y + 45 < 22.$$

Plugging Equation (1) it gives

$$24 < 6y \le 30. \tag{2}$$

Since (2) depends only of y, we must find the interval where y can take values. Clearly $y(max) = 5$, since P moves on C. To get $y(min)$, we intersect C with the lines AD (equation $x = 0$), and BC (equation $x = 6$) respectively, then we get $y = 4$. Therefore, $6y = 30$ for $y = 5$, and $6y = 24$ for $y = 4$. Because P must be inside the square ($0 < x, y < 6$) the value 30 can be taken (P is inside of the square), but the value 24 can not be reached (to be reached P must be on AD or BC).