Problem of the Week 14, Fall 2005

Solution by the organizers. Since the five points A, B, C, D, E are on a circle then, to prove that $ABCDE$ is a regular pentagon, it is enough to show that all sides have the same length.

The triangles ABC and BCD have the same area and they share $\overline{BC}$, thus the heights from A and D to the side BC have the same length. Therefore the line AD is parallel to BC. Thus $\angle DBC = \angle BDA$. On the other hand, by the Central Angle Theorem, we have that $\angle DBC = \angle DAC$. Then $\angle BDA = \angle DAC$ which proves that $ADBC$ is an isosceles trapezoid and consequently $AB = CD$. Similarly, starting with the pairs of triangles $(BCD, CDE), (CDE, DEA), (DEA, EAB), (EAB, ABC)$ we get $BC = DE$, $CD = EA$, $DE = AB$, and $EA = BC$. Thus $AB = CD = EA = BC = DE$ as we wanted to prove.

Would the conclusion still hold if the five points are not on a circle but they form a convex polygon? If not, How can these pentagons be characterized?