Problem of the Week 11, Fall 2005

Solution by the organizers. This solution does not require the concept of curvature. We first find the largest circle that touches the parabola at $(0,0)$. The center of such disk must be located at $(0,r)$ where r is the radius of the circle. Then the equation of this circle is

$$x^2 + (y-r)^2 = r^2.$$

To find the largest possible r we require that this circle intersects the parabola $y = x^2$ at a single point. Solving the system $y = x^2$ and $x^2 + (y-r)^2 = r^2$ we get

$$\begin{aligned} y + (y-r)^2 &= r^2 \\ y + y^2 - 2yr + r^2 &= r^2 \\ y\,(y + 1 - 2r) &= 0. \end{aligned}$$

Thus we get two solution, $y = 0$ and $y = 2r - 1$. However, y needs to be non-negative since $y = x^2$, thus to have only one solution we need $2r - 1 \le 0$. That is $r \le 1/2$. Thus $r = 1/2$ is the largest radius of a circle touching the parabola at $(0,0)$.

Now, let $x_0 \ne 0$ and (x_0, x_0^2) be a point on the parabola. By symmetry, the largest circle C on the concave part of the parabola touching the parabola at (x_0, x_0^2) must also touch the parabola at $(-x_0, x_0^2)$, and thus have a center on the y-axis. Suppose that the center of such circle is $(0,a)$. The slope of the tangent line to the parabola, at (x_0, x_0^2), equals the derivative $y' = 2x$ evaluated at x_0. Thus the slope of that line is $2x_0$. Since the segment joining $(0,a)$ and (x_0, x_0^2) is perpendicular to the tangent line at (x_0, x_0^2), we have that

$$-\frac{1}{2x_0} = \frac{x_0^2 - a}{x_0}.$$

Thus $a = x_0^2 + 1/2$. Let r be the radius of C, then we have that $x_0^2 + (x_0^2 - a)^2 = r^2$. Plugging the value of a we get $r^2 = 1/4 + x_0^2 > 1/4$. Therefore $r > 1/2$. Thus the circle of radius $1/2$ will freely rotate on the parabola, and $1/2$ is the radius of the largest such possible circle.

$y = x^2$