Problem of the Week 6, Fall 2005

Solution by the organizers. Assume first that $a \leq b \leq c$. Since each of a, b, and c divides $a+b+c$ then a divides $b+c$, b divides $a+c$, and c divides $a+b$. Note that $0 < a+b \leq 2c$ and since c divides $a+b$ then either $a+b=c$ or $a+b=2c$. Moreover $a+b=2c$ if and only if $a=b=c$. This gives the solution (a,a,a) for all positive integers a (note that in fact a divides $a+a+a=3a$).

Now assume $a+b=c$. Since b divides $a+c=2a+b$, then b divides $2a$. But $0<2a\leq 2b$ so $2a=b$ or $2a=2b$. If $2a=b$ then the triple $(a,2a,3a)$ is a solution for all positive integers a (note that a divides $a+2a+3a=6a$). If $2a=2b$, then the triple $(a,a,2a)$ is a solution for all positive integers a (note that a divides $a+a+2a=4a$).

Therefore all the solution triples are:

$$(a,b,c) \in \{(n,n,n),(n,2n,3n),(n,n,2n) : n \in \mathbb{N}\}.$$