Problem of the Week 3, Fall 2005

Solution by the organizers. Consider a graph G with six vertices representing each of the six irrational numbers. We color each of the edges connecting every pair of vertices according to the following rule. If x, y are vertices we color the edge xy *blue* if $x + y$ is rational, and we color it *red* if $x + y$ is irrational. In this context the problem asks us to prove that, regardless of the values of the six vertices, there is always a red triangle.

First we prove that it is impossible to have a blue triangle in G. Indeed, suppose x, y, z is a blue triangle. Then $x + y, y + z$, and $x + z$ are all rational numbers. By adding, subtracting, and multiplying rational numbers we get a rational number, thus

$$\frac{1}{2}\left((x + y) + (y + z) - (x + z)\right) = y$$

is a rational number. But this is a contradiction because y is an irrational number by assumption.

Pick an arbitrary vertex x_1, there are five edge coming out of x_1. By the pigeonhole principle at least three of these edges have the same color. Let x_1x_2, x_1x_3, and x_1x_4 be the edges with the same color c. We consider two cases. First, assume that c is red. Since there are no blue triangles, then at least one of the edges of the triangle $x_2x_3x_4$ is colored red, say (without loss of generality) x_2x_3. Then the triangle $x_1x_2x_3$ is colored red, which is what we wanted to prove. Now suppose c is blue. Since there are no blue triangles and the edges x_1x_2, x_1x_3, x_1x_4 are all colored blue, then the edges x_2x_3, x_3x_4, and x_2x_4 must be all colored red, which implies that the triangle $x_2x_3x_4$ is colored red. In both cases we prove that there is always a red triangle, that is there are always three numbers x, y, z, out of the given six, such that $x + y, x + z, y + z$ are all irrational.