California State University
Northridge

Problem of the Week

Department of Mathematics

Find the maximum value of

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)$$

over all triples of non-zero real numbers $a<b<c$ such that $a+b+c=0$.

Solution by George Craciun. Let $E=E(a,b,c)$ be the required expression. Then

$$\begin{aligned}
E &= 3+\left[\frac{b(b-c)}{a(c-a)}+\frac{c(b-c)}{a(a-b)}\right]+\left[\frac{a(c-a)}{b(b-c)}+\frac{c(c-a)}{b(a-b)}\right]+\left[\frac{a(a-b)}{c(b-c)}+\frac{b(a-b)}{c(c-a)}\right]\\
&= 3+([cb^2(b-c)^2(a-b)+c^2b(b-c)^2(c-a)]+\\
&\quad +[a^2c(c-a)^2(a-b)+ac^2(c-a)^2(b-c)]+\\
&\quad\quad +[a^2b(a-b)^2(c-a)+ab^2(a-b)^2(b-c)])/abc(a-b)(b-c)(c-a)\\
&= 3+(bc(b-c)^2\,[b(a-b)+c(c-a)]+\\
&\quad +ac(c-a)^2[a(a-b)+c(b-c)]+\\
&\quad\quad +ab(a-b)^2[a(c-a)+b(b-c)])/abc(a-b)(b-c)(c-a).
\end{aligned}$$

Since $a+b+c=0$ then,$b(a-b)+c(c-a)=ab+(c-b)(c+b)-ac=ab+(c-b)(-a)-ac=2a(b-c)$ and similarly, $a(a-b)+c(b-c)=2b(c-a)$ and $a(c-a)+b(b-c)=2c(a-b)$. Then

$$E=3+\frac{2abc[(b-c)^3+(c-a)^3+(a-b)^3]}{abc(a-b)(b-c)(c-a)}.$$

Let $x=a-b, y=b-c$, and $z=c-a$. Then $x+y+z=0$ and

$$E=3+\frac{2\left(x^3+y^3+z^3\right)}{xyz}$$

but, $x^3+y^3+z^3=(x+y+z)^3-3xy(x+y)-3z(x+y)(x+y+z)$. Then $x^3+y^3+z^3=-3xy(x+y)=-3xy(-z)=3xyz$, and

$$E=3+\frac{2\,(3xyz)}{xyz}=3+6=9.$$

[Editor's note: The expression is constant for all a,b,c such that none of the denominators is zero. Thus the maximum (and minimum) is 9]