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Find explicit formulas for all functions f, from the positive integers to the real numbers, such that

$$f(n) + f(m) = f(n)f(m) + f(n+m)$$

for all positive integers n and m.

Solution (Composite of John Foss and George Craciun). Let us call (*) the required condition. We claim that all functions satisfying (*) have the form $f(n) = 1 - c^n$ where c is an arbitrary real number.

We first verify that if $f(n) = 1 - c^n$ for every positive integer n, then f satisfies (*):

$$\begin{aligned} f(n)f(m) + f(n+m) &= (1-c^n)(1-c^m) + \left(1-c^{n+m}\right) \\ &= \left(1 - c^m - c^n + c^{n+m}\right) + 1 - c^{n+m} \\ &= (1-c^n) + (1-c^m) \\ &= f(n) + f(m). \end{aligned}$$

Now we prove that if f satisfies (*) then it must have the form $f(n) = 1 - c^n$ for some real number c. Let $c = 1 - f(1)$, if we use $m = 1$ and $n = N - 1$ in condition (*) we get

$$f(N-1) + f(1) = f(N-1)f(1) + f(N)$$

or equivalently

$$f(N) = (1 - f(1))f(N-1) + f(1) = cf(N-1) + (1-c).$$

For $N = 2, 3, \ldots$ we get

$$f(2) = cf(1) + (1-c) = c(1-c) + 1 - c = (1-c)(c+1),$$

$$\begin{aligned} f(3) &= cf(2) + (1-c) = c(1-c)(c+1) + (1-c) \\ &= (1-c)(c^2 + c + 1), \end{aligned}$$

$$\begin{aligned} f(4) &= cf(3) + (1-c) = c(1-c)(c^2 + c + 1) + (1-c) \\ &= (1-c)(c^3 + c^2 + c + 1), \end{aligned}$$

and in general

$$\begin{aligned} f(N) &= (1-c)(c^{N-1} + c^{N-2} + \cdots + c^2 + c + 1) \\ &= (1-c)\frac{(1-c^N)}{1-c} = 1 - c^N. \text{ (If } c \neq 1) \end{aligned}$$

If $c = 1$ then clearly $f(N) = 0$ for all positive integers N.