Problem of the Week. **September 6-13**

Proposed by Bernardo Ábrego and Silvia Fernández.

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with a, b, c, d real numbers and $A^3 = \begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}$. Find a, b, c, d.

Note: A^3 represents the **matrix multiplication** of A with itself three times.

Solution by Chuck Goodman.

Answer is : $\begin{pmatrix} 4/3 & 2/3 \\ 7/3 & -1/3 \end{pmatrix}$

First find the eigenvalues of A^3 and the corresponding eigenvectors and use these to diagonalize it.

The eigenvalues are 8, -1, so the corresponding eigenvectors are $[1,1]^T$ and $[2,-7]^T$

From this we see that $\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 7/9 & 2/9 \\ 1/9 & -1/9 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -7 \end{pmatrix}$

So, $\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 7/9 & 2/9 \\ 1/9 & -1/9 \end{pmatrix}$

and $\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}^{(1/3)} = \begin{pmatrix} 1 & 2 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} 8^{(1/3)} & 0 \\ 0 & -1^{(1/3)} \end{pmatrix}\begin{pmatrix} 7/9 & 2/9 \\ 1/9 & -1/9 \end{pmatrix}$

This simplifies to $\begin{pmatrix} 4/3 & 2/3 \\ 7/3 & -1/3 \end{pmatrix}$

and $\begin{pmatrix} 4/3 & 2/3 \\ 7/3 & -1/3 \end{pmatrix}^3 = \begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}$