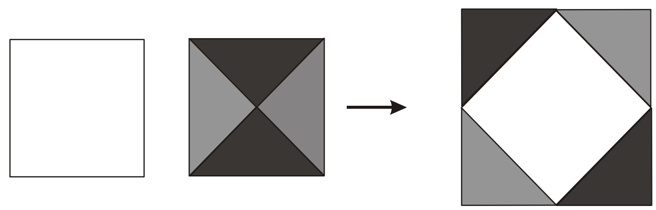
California State University
Northridge

Department of Mathematics

Problem of the Week

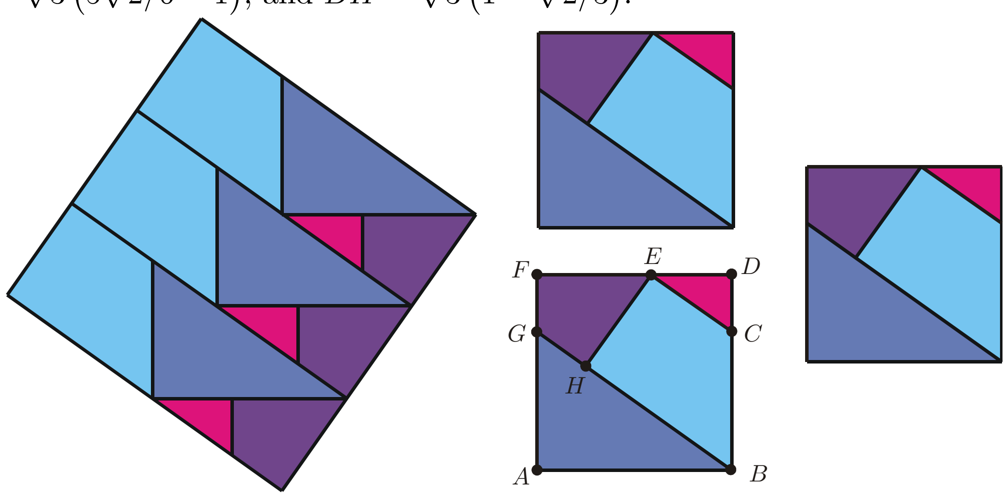
Proposed by Bernardo Ábrego and Silvia Fernández. **September 27-October 4**

The figure below shows how to cut two squares of area 1, so that the pieces can be rearranged to form a square of area 2.

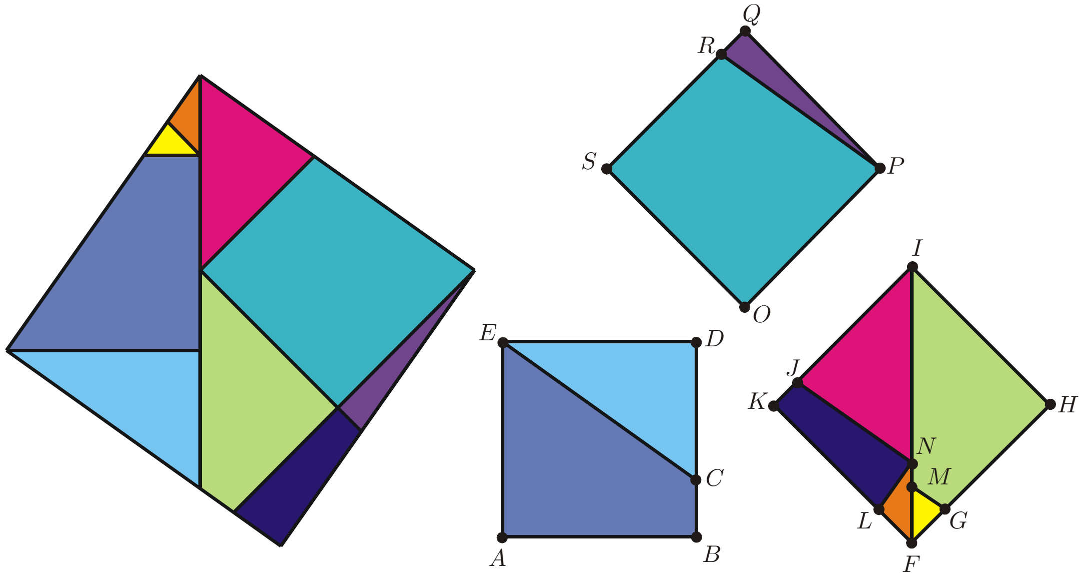


Show how to do the same with three squares. That is, show how to cut three squares of area 1 so that the pieces can be rearranged to form a square of area 3.
From all submitted solutions, the first one using the smallest number of pieces will be awarded. Multiple solutions are accepted (and encouraged).

Solution by W. Watkins. Twelve pieces are used. Note that each of the three unit squares is divided in the exact same way and in addition we only need to translate the pieces to obtain the square of area 3. The size of the pieces is given by: $AB = 1$, $BC = AG = \sqrt{2}/2$, $CD = FG = 1 - \sqrt{2}/2$, $CE = \sqrt{3}\left(1 - \sqrt{2}/2\right)$, $EH = \sqrt{3}/3$, $GH = \sqrt{3}\left(5\sqrt{2}/6 - 1\right)$, and $BH = \sqrt{3}\left(1 - \sqrt{2}/3\right)$.

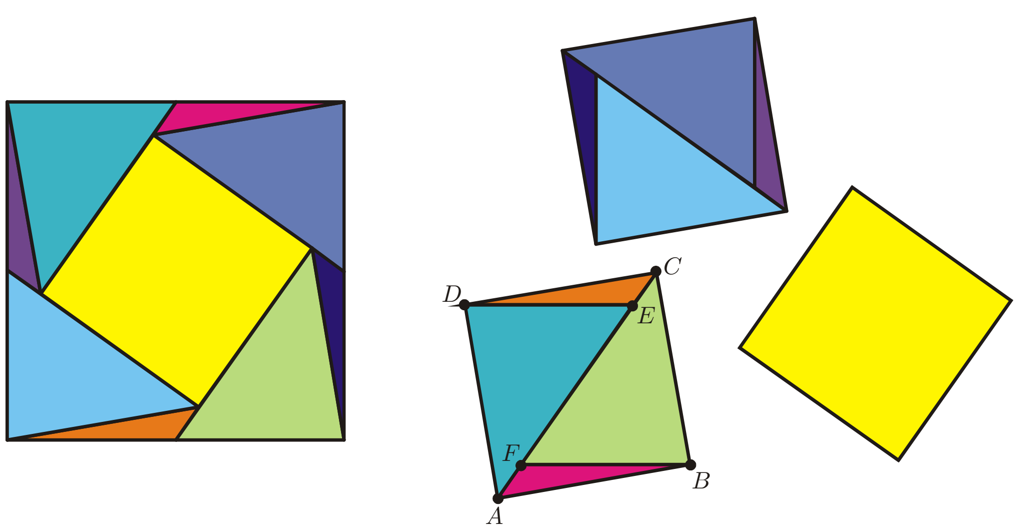


Solution 1 by B. M. Ábrego. Nine pieces are used. This solution is based on a classical proof of Pythagora's Theorem using dissection. The size of the pieces is given by: $AB = AE = OP = OS = PQ = HI = IM = 1$, $BC = FM = 1 - \sqrt{2}/2$, $BC = \sqrt{6}/2$, $QR = KJ = 3-2\sqrt{2}$, $RS = IJ = 2\sqrt{2}-2$, $FG = FL = 3\sqrt{2}-4$, $GH = KL = 5 - 3\sqrt{2}$, $JN = \sqrt{3}(\sqrt{2}-1)$, $LN = \sqrt{3}\left(3-2\sqrt{2}\right)$, $GM = \sqrt{3}\left(3\sqrt{2}/2-2\right)$, and $FN = \sqrt{2}-1$.



Solution 2 by B. M. Ábrego. Nine pieces are used. Note that, same as in the first solution, the pieces can be arranged to form the square of area 3 using only translations. Also, only three types of pieces are used (in fact we can divide the square in the center in the same way and then we would only have two types of pieces). Moreover, the eight pieces from the first two squares can be arranged to form a square of area 2, can you see how?

The size of the pieces is given by: $AB = BC = CD = AD = FE = 1$, and $AE = EC = (\sqrt{2}-1)/2$.



Solution 2 by S. Fernández. Only eight pieces are used! This is currently the best construction we know. It is not hard to see that at least 5 pieces are required. Thus the minimum number of pieces is between 5 and 8, and as far as we know this is an open question. The size of the pieces is given by: $AB = BC = CD = FG = GH = 1$, $DE = KJ = 1 - \sqrt{3}/3$, $AE = KF = KL = \sqrt{3}/3$, $JI = OP = ON = \sqrt{3} - 1$, $IH = IL = PQ = MN = 2 - \sqrt{3}$, $IK = 2 - 2\sqrt{3}/3$, $MT = QR = 2\sqrt{3} - 3$, and $ST = SR = 4 - 2\sqrt{3}$.