Problem of the Week. **September 6-13**

Proposed by Bernardo Ábrego and Silvia Fernández.

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with a, b, c, d real numbers and $A^3 = \begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}$. Find a, b, c, d.

Note: A^3 represents the **matrix multiplication** of A with itself three times.

Solution (by the organizers). We not only find a, b, c, and d but we actually prove there is a unique solution. First we observe that the characteristic polynomial of A^3 is

$$ch_{A^3}(x) = \det \begin{vmatrix} x-6 & 2 \\ 7 & x-1 \end{vmatrix} = x^2 - 7x - 8 = (x-8)(x+1).$$

We know then that $(A^3 - 8I)(A^3 + I) = 0$. Thus the minimal polynomial $m_A(x)$ of the matrix A should divide the polynomial

$$(x^3 - 8)(x^3 + 1) = (x-2)(x+1)(x^2 - x + 1)(x^2 + 2x + 4)$$

(The quadratic terms above cannot be factored over the reals).

Now, the degree of m_A is not one, otherwise A would be a multiple of the identity, and then clearly A^3 would not be equal to $\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}$. Thus the degree of m_A is two. Since m_A has real coefficients then, either $m_A(x) = (x-2)(x+1)$, or $m_A(x) = x^2 - x + 1$, or $m_A(x) = x^2 + 2x + 4$. If $m_A(x) = x^2 - x + 1$ then $m_A(A) = A^2 - A + I = 0$, and $0 = (A - I)(A^2 - A + I) = A^3 + I$ which is a contradiction. Similarly, if $m_A(x) = x^2 + 2x + 4$ then $0 = (A - 2I)(A^2 + 2A + 4I) = A^3 - 8I$ which is also a contradiction. Therefore $m_A(x) = ch_A(x) = (x-2)(x+1)$ and then -1 and 2 are the two eigenvalues of A.

Let v be an eigenvector of A with eigenvalue -1. Since $Av = -v$ then $A^3 v = -v$, so v is also an eigenvector of A^3 with eigenvalue -1. Similarly if u is an eigenvector of A with eigenvalue 2 then u is also an eigenvector of A^3 with eigenvalue 8. It is easy to see that $v = \binom{-2}{7}$ and $u = \binom{1}{1}$ work as some of these eigenvectors of A^3 (others can be used as well). Then since $Av = -v$ and $Au = 2u$ we get the system of equations

$$-2a + 7b = 2, \ -2c + 7d = 7, \ a + b = 2, \ c + d = 2$$

which yields the unique solution $a = 4/3$, $b = 2/3$, $c = 7/3$, and $d = -1/3$.