Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **November 8-15**

The positive integers p, p_1, p_2,and p_3 are prime numbers satisfying that $p_1 < p_2 < p_3$, and

$$p = p_1^2 + p_2^2 + p_3^2.$$

Prove that $p_1 = 3$.

Additional questions for possible projects.

Here is a very interesting project, that can definitely become a Master's Thesis:

> Prove or disprove that there are infinitely many primes p of the form $p = 9 + p_2^2 + p_3^2$ with p_2 and p_3 primes.

If we do not ask that p_2 and p_3 are primes then the answer is affirmative, in fact it is possible to exactly say which primes p can be written in the form $p = 9 + n_1^2 + n_2^2$. Also, if you fix one of the primes, say $p_2 = 5$, and ask if there are infinitely number of primes of the form $p = 9 + 25 + n_2^2 = 34 + n_2^2$, then the question is really difficult. For example it has been a long outstanding open problem to decide if there are infinitely many primes of the form $n^2 + 1$. (The answer should be yes).

So, by having two independent variables, p_2 and p_3, but requiring them to be prime, we may get a problem that can really be solved. Numerical values even suggest that perhaps the density of such numbers p may be approximated. For example, by trying all 66 pairs (p_2, p_3) with $p_2 < p_3$ primes in the range between the 25,000 prime and the 25,011 prime, we find that 12 of those pairs work. Here they are:

p_1	p_2	$p = 9 + p_1^2 + p_2^2$
287117	287191	164914842179
287117	287233	164938967987
287137	287149	164902204979
287137	287159	164907948059
287137	287167	164912542667
287137	287191	164926327259
287137	287219	164942410739
287137	287237	164952750947
287149	287191	164933218691
287159	287191	164938961771
287173	287219	164963085899
287179	287237	164976872219