Linear Equations

Section 1.1

- Demand as a function of price
- Supply as a function of price
- Solving linear equations
- Examples

Demand as a Function of Price

What happens to consumer demand if the price of a 52-inch Plasma HDTV goes down?

What happens to consumer demand if the price goes up?

Consumer demand is a *function* of the price. Symbols:

- Price: p is the selling price for the TV.
- Demand: d is the number of TVs (in thousands) sold.

Suppose that the quantity d of TVs sold (demanded) is related to the price p in a price-demand equation:

$$d = 1720 - .50p$$

$d = 1720 - .50p$ is an example of a linear equation in two variables. Why is the coefficient of p negative?

We are interested in pairs of numbers (p, d) that satisfy this equation. Such a pair is called a *solution* to the equation.

What is the demand (in thousands) if the price is \$1440?

What is the demand if the price is \$2500?

What price should we charge if we want to sell 500 thousand TVs?

You found that the demand (in thousands) if the price is \$1440 is 1000 thousand TVs (one million). What is the demand if the price is \$1441?

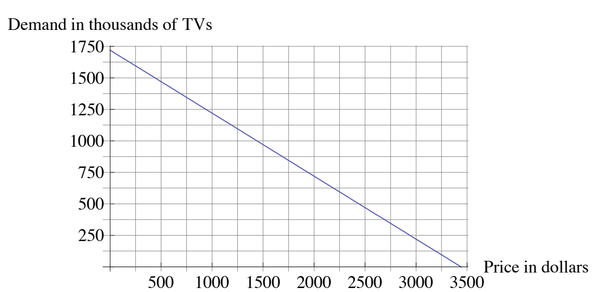
The demand (in thousands) if the price is \$2500 is

$$d = 1720 - (.50)(2500) = 470 \text{ thousand TVs.}$$

What is the demand if the price is \$2501?

If price increases by \$1, how does demand change?

The graph below represents all solutions of $d = 1720 - .50p$.



Mark an x at the point above that represents the situation where the demand is 500 thousand TVs. Estimate the corresponding price.

Rewrite the price-demand equation, $d = 1720 - .50p$, by solving for p in terms of d.

In what circumstances would you use this equation rather than the original?

Supply as a Function of Price

Companies that make the 52-inch HDTVs are willing to produce more TVs if they can sell them at a higher price. So the supply of TVs is also a function of the price at which these TVs will sell. Suppose that the supply equation is:

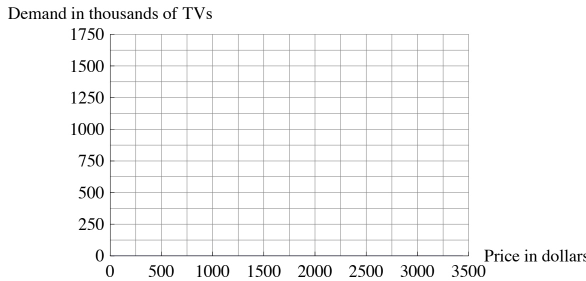
$$s = .375p + 460,$$

where s is the supply measured in thousands of TVs.

Why is the coefficient of p now positive?

Is this also a linear equation in two variables?

Make a graph that represents all solutions of the price-supply equation: $s = .375p + 460$ (with scales and labels) for prices less than \$3500.



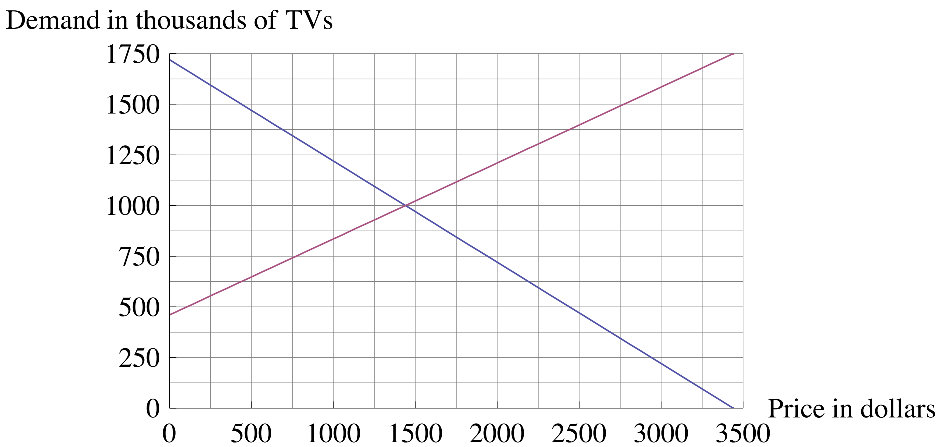
Use the graph to estimate:

- the supply if the price is \$600

- the supply if the price is \$2500

Is there a price where the quantity demanded is equal to the quantity supplied? This is called the *equilibrium point*. Price-demand equation: $d = 1720 - .50p$
Price-supply equation: $s = .375p + 460$

This also can be solved graphically:



Simple Interest Formula

Symbols:

- Principal: P
- Interest rate (per time period): r
- Number of time periods: t
- Amount of money at end of investment period: A

These also are related by a linear equation: $A = P + Prt$. Find the amount at the end of 4 years when \$5,000 is invested at 3% simple interest per year.

Find the principal that would yield an amount of \$5,000 at the end of 4 years at 3% simple interest per year.

Solve $A = P + Prt$ for each variable below.

Solve for r

Solve for P

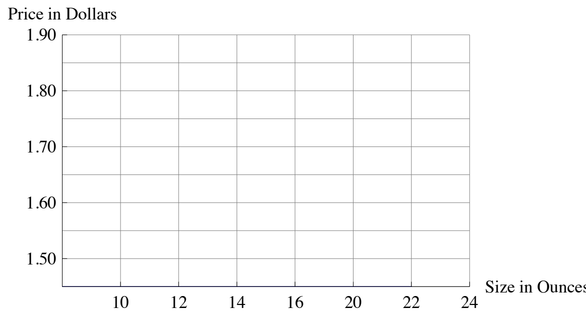
More Practice with Linear Situations: Price of coffee

1. The prices for a cup of coffee in the Freudian Sip are given in the table below.

Size in Ounces	Price in $
12	1.55
16	1.75
20	1.85

a. Is the relationship between size and price linear? How can you tell from the table?

b. Graph the three size/price points on the graph.



b. How can you tell from the graph if the relationship between size and price is linear?

c. To make the relationship linear, what should a 20 ounce cup cost?

Size in Ounces	Price in $
12	1.55
16	1.75
20	1.85

d. Alternatively, what should a 16 ounce cup cost?

2. Mary paid a total of $28,400 for a new car. This total included the purchase price, 8.5% sales tax, and a $190 title and license fee. What was the purchase price of the car? [Similar to #8 Section 1.1]

3. A company makes CDs. It spends $18,000 each year for fixed costs (building, utilities, etc.). The variable cost (cost of materials, labor, etc.) is $5.20 per CD. Each CD sells for $7.60. How many CDs, x, would the company have to make and sell to break even? [Similar to #9 Section 1.1]

Linear Equations: Slope, Intercepts, Applications

Section 1.2

- Linear Equations in two variables
- Slope of a line
- Intercepts
- $y = mx + b$
- Applications

$x-$ and $y-$ intercepts of an equation:

Definition In the graph of any equation of two variables, the points where the graph of an equation crosses the x-axis are called the x-**intercepts** and the points where the graph crosses the y-axis are called the y-**intercepts**.

What do the coordinates of an x-intercept look like?
To find them set $\quad = 0$ and solve for $\quad$.

What do the coordinates of a y-intercept look like?
To find them set $\quad = 0$ and solve for $\quad$.

The intercepts of a linear equation are easier to locate than the intercepts of most other equations.

x- and y-intercepts of a linear equation:

Find the x- and y- intercepts for $4x - 3y = 12$.

Find the x- and y-intercepts for $7x - .2y = 12$.

Could a linear equation have more than one x-intecept?

Intercepts for the Price-demand equation

$$d = 1720 - .50p$$

What are the p- and d-intercepts? What do they mean here?

The slope of a linear equation: the price-demand equation

$$d = 1720 - .50p$$

If price increases by \$1 how much does demand decrease?

Does this depend on the starting price?

If price increases by \$1000 how much does demand decrease?

Does this depend on the starting price?

Is the decrease in demand *always* .50 times the increase in price?

Slope-Intercept form ($y = mx + b$)

Definition: An equation of the form

$$y = mx + b$$

is a linear equation in **slope-intercept form**.

Put $4x - 3y = 12$ into slope-intercept form.

What are m and b?

Give a verbal description of m and b.

Price-demand equation: $d = 1720 - .50p$.

Put this into slope-intercept form.

What are m and b?

What is the meaning of m? of b?

Application: Depreciation

Linear Depreciation. Office equipment was purchased for \$20,000 and is assumed to have a scrap value of \$2,000 after 10 years. If its value is depreciated linearly (for tax purposes) from \$20,000 to \$2,000:

1. What is the slope of the line? Write a verbal interpretation of the slope of the line.

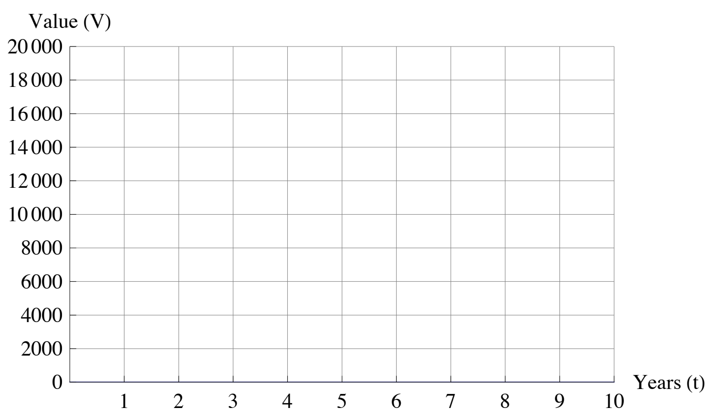
2. Find the linear equation that relates value (V) in dollars to time (t) in years. (Hint: you know two points.)

Application: Depreciation

Linear Depreciation. Office equipment was purchased for \$20,000 and is assumed to have a scrap value of \$2,000 after 10 years. If its value is depreciated linearly (for tax purposes) from \$20,000 to \$2,000:

1. What would be the value of the equipment after 6 years?

2. Graph the equation $V = -1800t + 20000$ for $0 \leq t \leq 10$



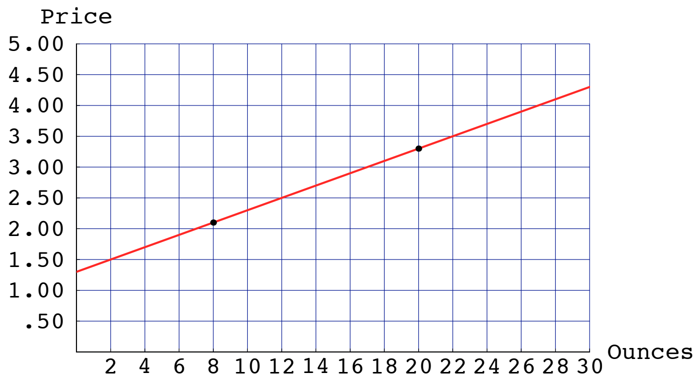
Application: Linear Interpolation

The price of a cup of coffee in a coffee bar depends on the size of the cup. The 8-ounce cup costs \$2.10, but the larger 20-ounce cup costs \$3.30. Without any other information, how could you estimate the cost of a 10-ounce cup, or a 16-ounce cup?

Sometimes business people use a method known as linear interpolation. This means that they **assume** that the price p of a cup of coffee and the size q of the cup obey a **linear** equation. So the graph of the linear equation is a line and we know two points on this line: $(q, p) = (8, 2.10), (20, 3.30)$.

Application: Linear Interpolation

two points on the line: $(q, p) = (8, 2.10), (20, 3.30)$.



Slope: $m = \frac{3.30-2.10}{20-8} = \frac{1.20}{12} = .10$
Point-slope form: $p - 2.10 = .10(q - 8)$
Slope-intercept form: $p = .10q + 1.30$.

Application: Linear Interpolation

Slope-intercept form: $p = .10q + 1.30$.

Use linear interpolation to find the price of a 12-ounce cup of coffee:

Give a verbal interpretation for the slope:

If the size of the cup increases by 4 ounces, by how much does the cost increase?

Give a verbal interpretation for the y-intercept:

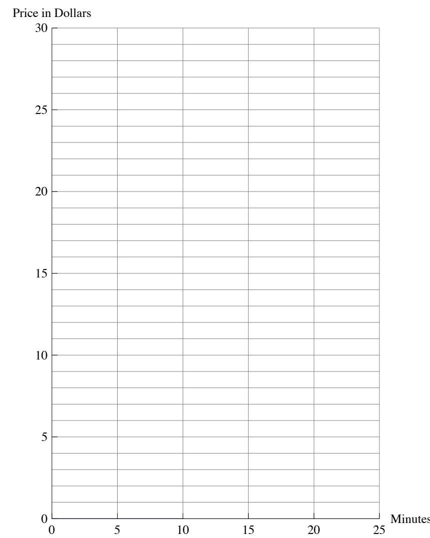
Application: So Relax

Westfield shopping mall has an accupressure massage station called "So Relax." Here is the pricing structure:

10 minutes \$13.00
15 minutes \$19.00
20 minutes \$26.00

- Plot these prices
- Is the price a linear function of the time in minutes?

- What is the best deal?



Functions, Part 1
Section 2.1

- Notation
- Evaluation
- Solving
- Unit of measurement

Introductory Example: Fill the gas tank

Your gas tank holds 12 gallons, but right now you're running on empty. As you pull into the gas station, the engine sputters and dies—the gas tank is completely empty. You pump 12 gallons into the tank and swipe your credit card. How much does it cost?

That depends on the price of gas, of course! But exactly how does it depend on the price?

If the price per gallon is \$3.40, what is the cost to fill the tank?

If the price per gallon is \$4.10, what is the cost to fill the tank?

Fill the gas tank: function and symbolic form

The symbolic way to state exactly how the cost to fill the tank depends on the price per gallon is to write an algebraic expression for the cost in terms of the price. In our example:

$$C(p) = 12p.$$

Terminology (function): The cost, $C(p)$ to fill the tank is a **function** of the price p per gallon.

The symbolic form of the statement "the cost of filling the tank at a price of \$3.40 per gallon is \$40.80," is $C(3.40) = 40.80$.

Our purpose here is to practice translating statements in words into symbolic form using function notation.

Fill the gas tank

Here are some other examples from the gas tank situation:

Words: At a price of \$2.00 per gallon, it costs \$24.00 to fill the tank.
Write the symbolic form:

Words: The cost to fill a 12-gallon tank at a price of p dollars per gallon is 12 times p.
Write the symbolic form:

Words: What is price per gallon if it costs \$45.00 to fill the tank?
Symbolic form: What is p if $C(p) = 45.00$?
Solve $12p = 45.00$ for p.

Evaluate a function

Evaluate the function $C(p) = 12p$ at $p = 3.50$.
Plug $p = 3.50$ into the expression $12p$.
Given the input $p = 3.50$, determine the output $C(p)$.

$C(3.50) = 12 \times 3.50 = \$42.00$

Evaluate the function $C(p) = 12p$ at $p = 2.00$

Evaluate the function $C(p) = 12p$ at $p = 3.00$

Solve an equation

Given a cost $C(p)$, say $C(p) = \$45.00$, find the price p at which the cost is \$45.00.
Which value of p can you plug into the expression $12p$ so that $12p = 45.00$?
Given the output $C(p) = \$45.00$, what is the input p?

If we want to know what price per gallon results in a cost of \$45.00 to fill the tank, we must solve $12p = 45.00$ for p:

$$\begin{aligned} 12p &= 45.00 \\ p &= 45.00/12 \\ &= 3.75. \end{aligned}$$

Summary: If the cost to fill the tank is \$45.00, then the price per gallon is \$3.75.

Units of measurement

In the gas tank example, the price of gas p is measured in dollars per gallon and the cost to fill the tank $C(p)$ is measured in dollars. These are the **units of measurement** in this example. In other business examples the units of measurement could be money measured in thousands of dollars, number of television sets produced, square feet in a house, or kilowatt hours. It all depends on the business setting. You should always state your results using the proper units of measurement.

Goals:

1. use function notation to make statements about business situations

2. use functional notation to solve problems in business settings

3. summarize the results of our symbolic (algebraic) manipulations into statements in a business setting

4. use proper units of measurement for the functions and statements.

Electricity costs

Edcon power company charges its residential customers \$14.00 per month plus \$0.10 per kilowatt-hour (KWH) of electricity used. Thus, the monthly cost for electricity is a function of the number of KWHs used. In symbols, let k be the number of KWHs used in a month, and $E(k)$ be the monthly cost for electricity in dollars.

- What are the units of measurement for k and for $E(k)$?

- Write the symbolic form for the statement: The monthly cost for using 800 kilowatt-hours of electricity is \$94.00.

Electricity costs

Edcon power company charges its residential customers \$14.00 per month plus \$0.10 per kilowatt-hour (KWH) of electricity used. Thus, the monthly cost for electricity is a function of the number of KWHs used. In symbols, let k be the number of KWHs used in a month, and $E(k)$ be the monthly cost for electricity in dollars.

- Write the symbolic statement $E(660) = 80$ in words.

- Write the symbolic form for the statement: The monthly cost for using k KWHs is \$100.00.

Continuing with the electricity costs, we can give a formula for the electricity costs as a function of kilowatt-hours uses as follows:

$$E(k) = 14.00 + 0.10k. \tag{1}$$

The 14.00 in the formula is the amount the amount of money the customer pays regardless of how many kilowatt-hours are used. Even if the customer uses no electricity, the monthly charges will still be \$14.00. This is called the **fixed cost**. In addition to the fixed cost, the customer pays \$0.10 per kilowatt-hour used. So k KWHs would cost an additional \$0.10k. Adding these two types of costs, we get a total cost of $14.00 + 0.10k$ dollars per month.

$$E(k) = 14.00 + 0.10k.$$

If the customer uses 200 KWHs, find the cost.

Summary:

One of the reasons for writing statements in symbolic form is that it permits us to manipulate algebraic expressions in order to solve problems. For example,

How many KWHs can be used if the monthly cost is \$55.00?

In this case, we do not know k, the number of KWHs used. But we do know the cost of using k KWHs, $E(k) = 55.00$. So by replacing $E(k)$ with the formula (1) we can write $E(k) = 55.00$ as

$$14.00 + 0.10k = 55.00.$$

Now the problem is to solve the above equation using algebra.

Summary:

Problems on electricity costs

$$E(k) = 14.00 + 0.10k$$

Translate each problem into symbolic form, solve, and write a summary.

- What is the monthly cost for using 600 KWHs?

- How many KWHs can be used if the monthly cost is \$50.00?

Functions, Part 2

Section 2.1

- Function notation
- Definition
- Graphs
- General cost and demand functions

Function Notation

$$y = 2x - 1 \qquad f(x) = 2x - 1$$

$f(2) \quad =$

$f(-1) =$

$f(0) \quad =$

$f(2/3) =$

x, y equation	function
$y = -5x + 3$	$f(x) = -5x + 3$
$y = 2x^2 + 4x - .5$	$g(x) = 2x^2 + 4x - .5$

$f(-3) \quad =$ $\qquad\qquad$ $g(1) \quad =$

$f(a) \quad =$ $\qquad\qquad$ $g(a) \quad =$

$f(a+1) =$ $\qquad\qquad$ $g(a+2) =$

Definition of a Function

A **function** is a rule that produces a correspondence between two sets of numbers such that each number in the first set (called the **domain** of the function) corresponds to exactly one number in the second set (called the **range** of the function).

Example:

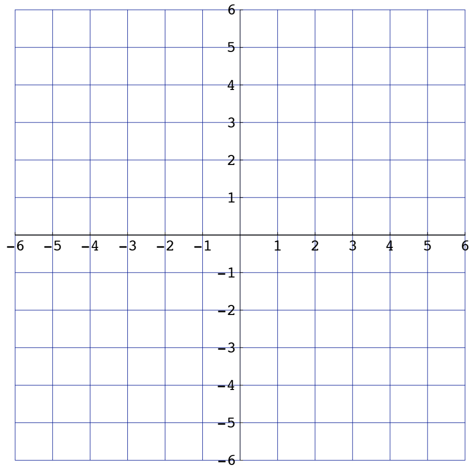
$$E(k) = 14.00 + 0.10k$$

The domain is:

The range is:

Example:

$$y = \sqrt{x}$$



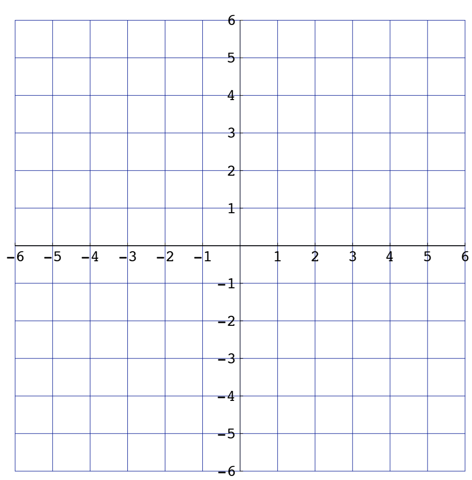
The domain is:

Inequality notation:

Interval notation:

Example:

$$y = \frac{1}{x}$$



The domain is:

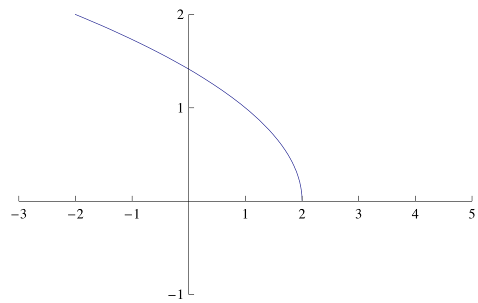
Inequality notation:

Interval notation:

Exercise: Find the Domain

Example:

$f(x) = \sqrt{2-x}$



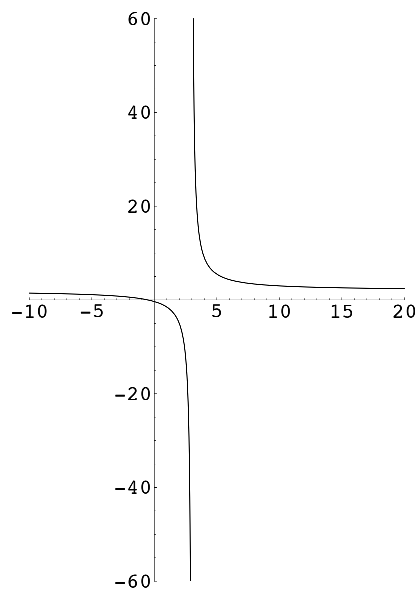
Domain of $f(x)$:

Inequality notation:

Interval notation:

Exercise: Find the Domain

Example: $f(x) = \dfrac{2x+1}{x-3}$



Domain of $f(x)$:

The expression makes sense for all values of x EXCEPT $x = 3$.

Inequality notation: $x \neq 3$

Cost, Price-Demand, Revenue, and Profit Functions

x is the number of units sold (independent variable)

Cost Function: $C(x) = a + bx$

$C(x)$ is the cost to produce x units.
There is a fixed cost, a and a cost per unit, b.

Price-demand Function: $p(x) = m - nx$

$p(x)$ is the price per unit. The price per unit decreases as the number of units produced increases.

Revenue Function: $R(x) = xp(x) = x(m - nx) = xm - nx^2$

Revenue is the number sold times the price per unit.

Profit Function: $P(x) = R(x) - C(x)$

Profit is revenue minus cost.

Example: Publishing a book

A publishing company is preparing to market a new book on turnip gardening. The author gets a flat fee of $5000 for writing the book. It costs $8200 to to typeset the book plus $18 per book to print and bind the book.

- Find and interpret $C(x)$.

- What is the cost to publish 2000 books?

- How many books can be published for $40,000?

Publishing a book, continued

The publisher believes that price, p, as a function of demand, x, is $p(x) = 140.00 - 0.10x$.

- Find the revenue function $R(x)$.

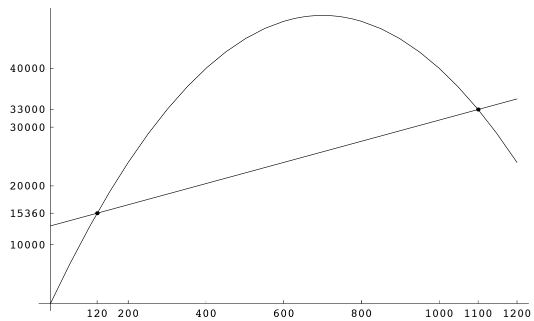
- Find the profit function $P(x)$.

Publishing a book, answers

Here are the correct functions:

$C(x) = 13200 + 18x$	$p(x) = 140.00 - .10x$
$R(x) = 140x - .10x^2$	$P(x) = -13200 + 122x - .10x^2$

Break-even points



Break-even points occur when cost equals revenue. How many units must be produced to break-even?

Solve $C(x) = R(x)$ for x or equivalently, $P(x) = 0$.

Elementary Functions and Transformations

Section 2.2

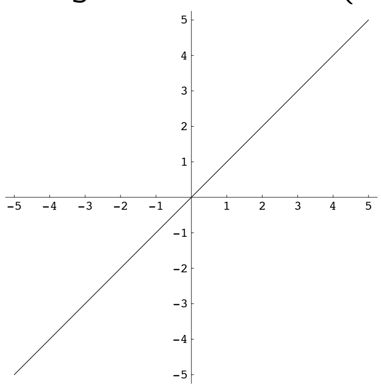
- A beginning library of elementary functions
- Graphs of elementary functions
- Shifts and stretches
- Piecewise -defined functions

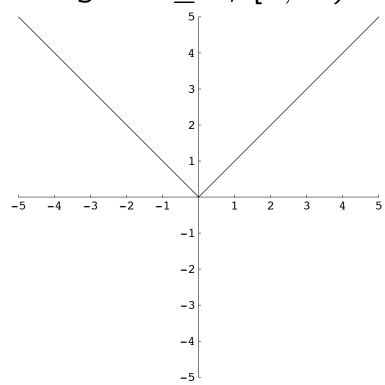
Large Beginning Library

- identity function $f(x) = x$
- absolute value function $f(x) = |x|$
- square function $f(x) = x^2$
- square-root function $f(x) = \sqrt{x}$
- piecewise defined functions

Identity and Absolute value functions

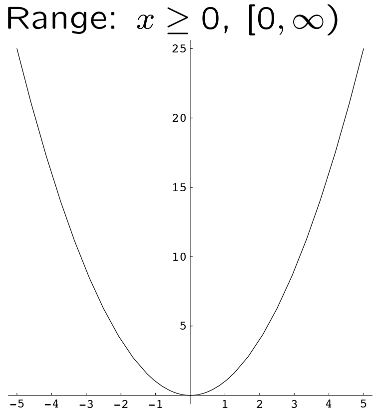
Identity function	Absolute-value function
Expression $f(x) = x$	Expression $f(x) = \lvert x \rvert$ or abs (x)
Domain: all numbers $(-\infty, \infty)$	Domain: all numbers $(-\infty, \infty)$
Range: all numbers $(-\infty, \infty)$	Range: $x \geq 0$, $[0, \infty)$

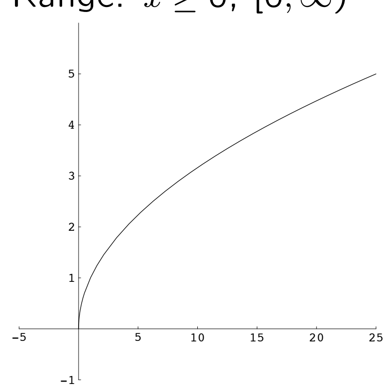




Square and Square-root functions

Square function	Square-root function
Expression: $f(x) = x^2$	Expression: $f(x) = \sqrt{x}$
Domain: all numbers $(-\infty, \infty)$	Domain: $x \geq 0$, $[0, \infty)$
Range: $x \geq 0$, $[0, \infty)$	Range: $x \geq 0$, $[0, \infty)$



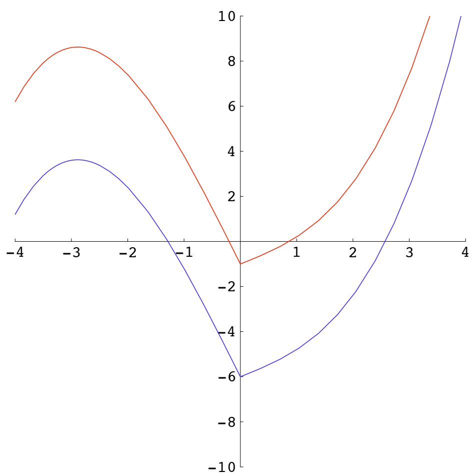


Transformations

- vertical translations (shift)
- vertical stretch
- horizontal translation (shift)

Vertical shift

Vertical shift 5 units up



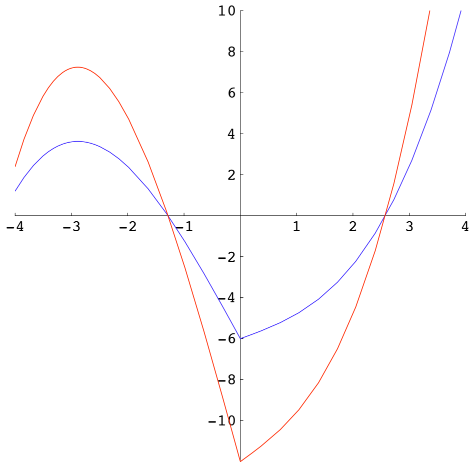
The graph of $f(x)$ is blue (dark line).

The graph of $f(x) + 5$ is red (light line).

The vertical distance between the curves is 5.

Vertical stretch

Vertical stretch by a factor of 2:



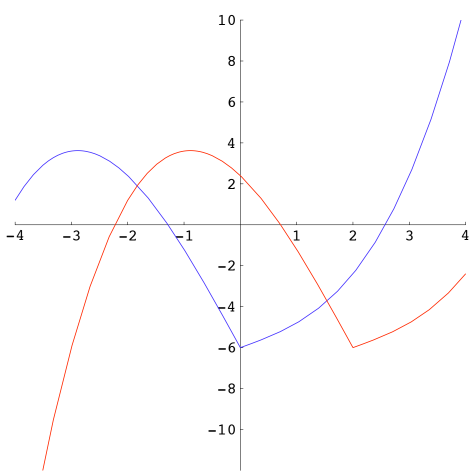
The graph of $f(x)$ is blue (dark line).

The graph of $2f(x)$ is red (light line).

The vertical distance from the x-axis of the graph of $2f(x)$ is twice that of $f(x)$.

Horizontal shift

Horizontal shift two units to the right



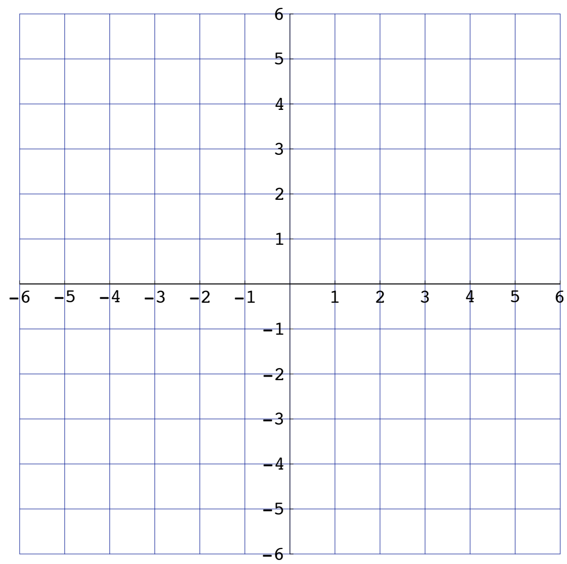
The graph of $f(x)$ is blue (dark).

The graph of $f(x-2)$ is red (light).

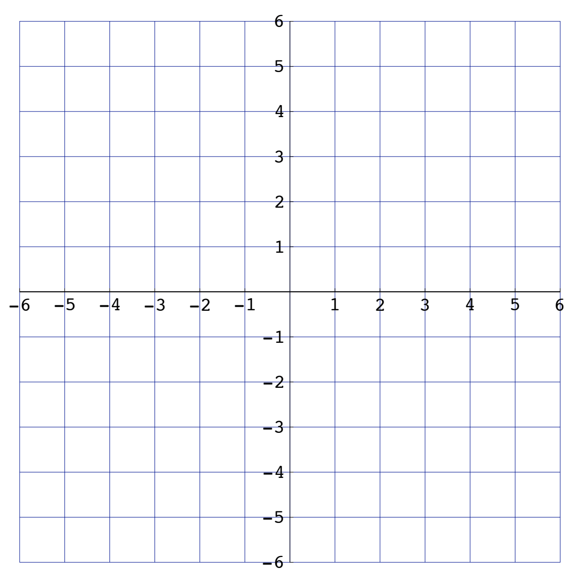
The horizontal distance between the curves is 2.

Practice: Graph these functions

$y = 2|x|$



$y = |x + 3|$



Practice: Each function corresponds to geometric description

$f(x-5)$	horizontal shift 5 units to the right
$f(x)+7$	
$3f(x)$	
$f(x-3)-1$	
	vertical shift 2 units up
	vertical shrink by a factor of 1/2
	horizontal shift 4 units to left

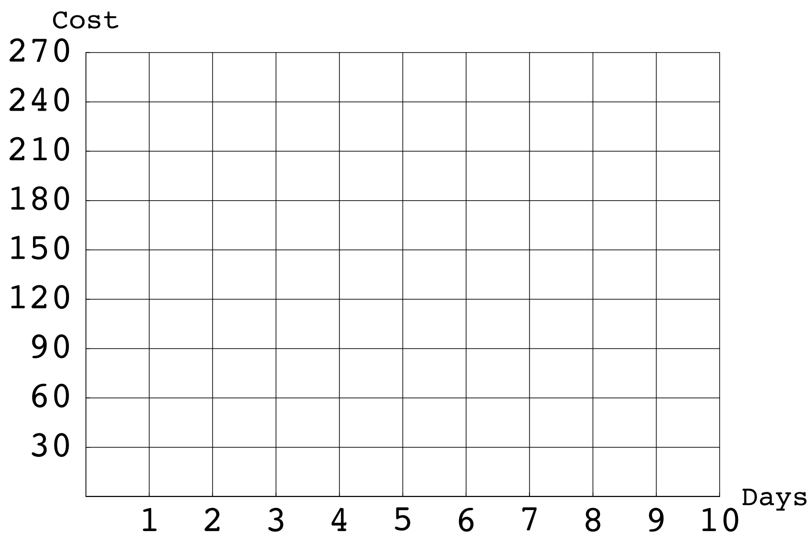
Piecewise defined functions, an example

A car rental agency charges \$30 per day (or partial day) or \$150 per week, whichever is least. What is the rental cost $C(x)$ for x days?

Here are the costs for various numbers of days, x. Fill in the two missing costs.

x	1.0	2.0	2.6	3.0	3.1	4.0	4.2	5.0	6.0	7.0	7.1
C(x)		60	90	90		120	150	150	150	150	180

A car rental agency charges \$30 per day (or partial day) or \$150 per week, whichever is least. Graph the cost function $C(x)$.



Example from business $T(x)$ is the tax on taxable income of x.

The federal income tax rate is

Between	But Not Over	Base Tax	Rate	Of the Amount Over
\$0	\$7,550	0	10%	\$0.00
\$7,550	\$30,650	\$755.00	15%	\$7,550
\$30,650	\$74,200	\$4,220.00	25%	\$30,650
\$74,200	\$154,800	\$15,107.50	28%	\$74,200
\$154,800	\$336,550	\$37,675.50	33%	\$154,800
\$336,550		\$97,653.00	35%	\$336,550

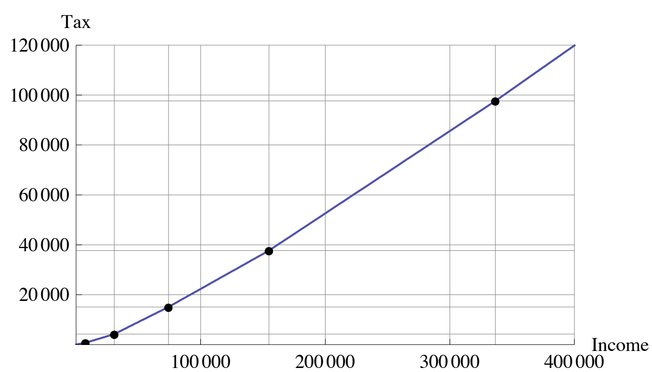
If you have a taxable income of $x = \$110,000$, your tax is

$$
\begin{aligned}
T(110,000) &= \text{Base Tax} + (\text{Rate} \times \text{Amount Over}) \\
&= 15,107.50 + [.28 \times (110,000 - 74,200)] \\
&= 15,107.50 + [.28 \times 35,800] \\
&= 15,107.50 + 10,024.00 \\
&= 25,131.50
\end{aligned}
$$

Between	But Not Over	Base Tax	Rate	Of the Amount Over
\$0	\$7,550	0	10%	\$0.00
\$7,550	\$30,650	\$755.00	15%	\$7,550
\$30,650	\$74,200	\$4,220.00	25%	\$30,650
\$74,200	\$154,800	\$15,107.50	28%	\$74,200
\$154,800	\$336,550	\$37,675.50	33%	\$154,800
\$336,550		\$97,653.00	35%	\$336,550

Suppose you have a taxable income of $x =$\$50,000. What is your tax?

The graph of $T(x)$:



- What is the slope of the line segment between 74,200 and 154,800?

- What is the height of the function when income is \$74,200?

- What are the coordinates of the point shown at the upper right of the graph?

The equations for $T(x)$:

Between	But Not Over	Base Tax	Rate	Of the Amount Over
\$0	\$7,550	0	10%	\$0.00
\$7,550	\$30,650	\$755.00	15%	\$7,550
\$30,650	\$74,200	\$4,220.00	25%	\$30,650
\$74,200	\$154,800	\$15,107.50	28%	\$74,200
\$154,800	\$336,550	\$37,675.50	33%	\$154,800
\$336,550		\$97,653.00	35%	\$336,550

For income between \$74,200 and \$154,800:
Line 4 in the table.
$74200 \leq x \leq 154800$: $T(x) =$

The equations for $T(x)$:

Between	But Not Over	Base Tax	Rate	Of the Amount Over
\$0	\$7,550	0	10%	\$0.00
\$7,550	\$30,650	\$755.00	15%	\$7,550
\$30,650	\$74,200	\$4,220.00	25%	\$30,650
\$74,200	\$154,800	\$15,107.50	28%	\$74,200
\$154,800	\$336,550	\$37,675.50	33%	\$154,800
\$336,550		\$97,653.00	35%	\$336,550

For income between \$30,650 and \$74,200:
Line 3 in the table.
$30650 \leq x \leq 74200$: $T(x) =$

Income at the end of an income bracket

What is the tax on an income of exactly \$74,200? Which line in the table should be used–line 3 or line 4?

Between	But Not Over	Base Tax	Rate	Of the Amount Over
\$0	\$7,550	0	10%	\$0.00
\$7,550	\$30,650	\$755.00	15%	\$7,550
\$30,650	\$74,200	\$4,220.00	25%	\$30,650
\$74,200	\$154,800	\$15,107.50	28%	\$74,200
\$154,800	\$336,550	\$37,675.50	33%	\$154,800
\$336,550		\$97,653.00	35%	\$336,550

Quadratic Functions and Parabolas

Section 2.3

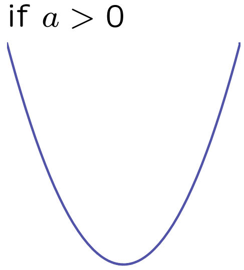
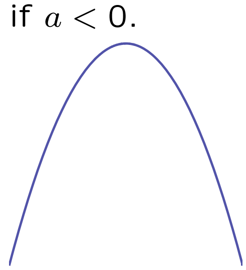
- Parabolas
- Quadratic equations and functions
- Graphs of quadratic functions
- Applications

Quadratic Functions and Expressions

The standard form for a quadratic function is

$$f(x) = ax^2 + bx + c.$$

The graph of a quadratic function is a parabola.

The parabola opens up if $a > 0$	The parabola opens down if $a < 0$.
	

Vertex of the parabola, maximum and minimum

$$f(x) = ax^2 + bx + c.$$

The vertex of the parabola is found at the point where $x = -b/2a$.

Often the vertex is denoted (h, k). In this case, $h = -b/2a$ and k can be determined by the equation.

Example:

What are the coordinates of the vertex of the graph of $f(x) = 3x^2 + 6x - 5$? $(h, k) = (\quad , \quad)$

Vertex-axis form for a quadratic function

- $f(x) = ax^2 + bx + c$ (standard form)
- $f(x) = a(x - h)^2 + k$ (vertex-axis form)

Since

$$a(x - h)^2 + k = ax^2 - 2ahx + ah^2 + k,$$

$$b = -2ah, \text{ or } h = -b/2a.$$

Once you know the x-coordinate (called h) of the vertex, the y-coordinate (called k) is

$$k = f(h).$$

Example and Exercise:

Remember: $h = -b/2a$.

Find the coordinates of the vertex of the graph of

$$f(x) = 2x^2 + 12x + 13.$$

$a = 2, b = 12, c = 13$, so $h = -3$ and then
$k = f(-3) = 2(-3)^2 + 12(-3) + 13 = -5$.

So the coordinates of the vertex are $(-3, -5)$.

Next find the vertex-axis form of the quadratic function $f(x)$.

The vertex-axis form is

$$f(x) = a(x-h)^2 + k = 2(x+3)^2 - 5.$$

Check it:

$$2(x+3)^2 - 5 = 2(x^2 + 6x + 9) - 5 = 2x^2 + 12x + 13.$$

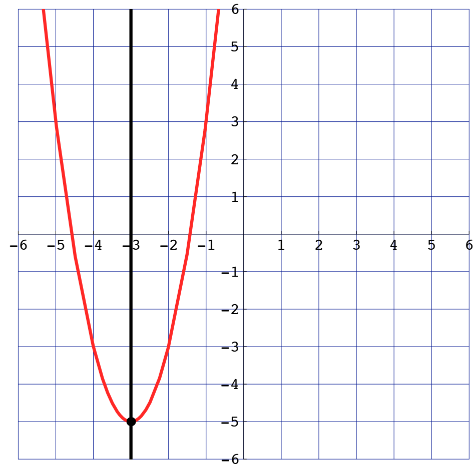
Exercise:

Find the coordinates of the vertex and convert the standard form into the vertex-axis form:

$$f(x) = -x^2 + 3x - 7.$$

Graphing a quadratic function

$$y = 2x^2 + 12x - 5$$



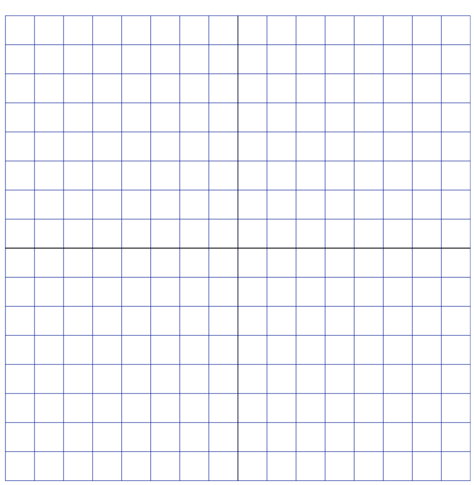
The vertex is the point at

The axis of symmetry is the vertical line

Exercise:

Graph the parabola:

$f(x) = x^2 - 6x + 10$



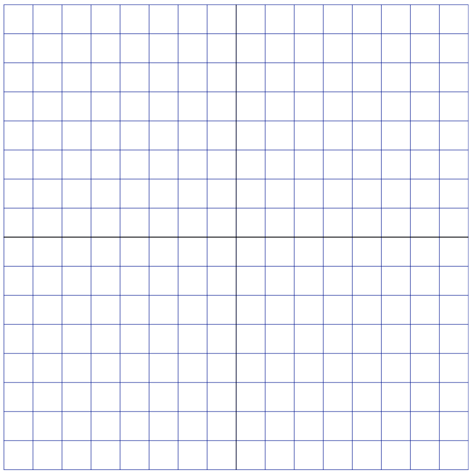
Opens:

Vertex:

y-intercept:

Exercise:

Graph the parabola:
$f(x) = 3x^2 + 6x + 1$



Opens:

Vertex:

y-intercept:

The quadratic formula

The general quadratic function:

$$f(x) = ax^2 + bx + c$$

The quadratic formula tells you the solutions to $f(x) = 0$, which is the same as locating the x-intercepts on the graph:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example: Solve

$$2x^2 - 5x - 3 = 0,$$

for x.

$$a = 2, \quad b = -5, c = -3$$

$$x = \frac{5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} = \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4}.$$

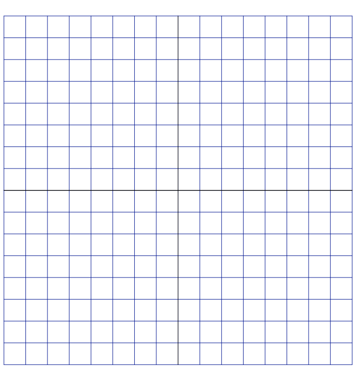
So $x = 3$ and $x = -1/2$ are the solutions.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Exercise: Graph $y = x^2 - 5x - 6$ and solve $x^2 - 5x - 6 = 0$, for x.

$$a = \qquad , \quad b = \qquad , c =$$



Summary: Quadratic Functions

$$f(x) = ax^2 + bx + c,$$

where a is not equal to zero

- if $a > 0$, the graph opens ______
- if $a < 0$, the graph opens ______
- x-coordinate of vertex $h =$
- $f(h) = k$ is minimum if $a > 0$
- $f(h) = k$ is maximum if $a < 0$
- Domain: set of all real numbers
- Range: $\begin{array}{ll} (-\infty, k], & \text{if } a < 0 \\ [k, \infty)\,, & \text{if } a > 0 \end{array}$

Rational Functions and Polynomials

Section 2.3

Rational Functions:	• graphs • asymptotes
Polynomial Functions:	• turning points • roots

Rational functions

A rational function is a function of the form

$$f(x) = \frac{n(x)}{d(x)},$$

where both the numerator, $n(x)$, and the denominator, $d(x)$ are polynomials.

These are rational functions:

$$\begin{aligned}
f(x) &= \frac{1}{x} \\
f(x) &= \frac{2x-3}{x+8} \\
f(x) &= \frac{29x^{19}+77x^2-89}{x^{16}-55x^9+44} \\
f(x) &= \frac{x+26}{4x^2-7x+22}
\end{aligned}$$

Simple rational functions

We will study only one simple type of rational function, those of the form

$$f(x) = \frac{ax+b}{cx+d}.$$

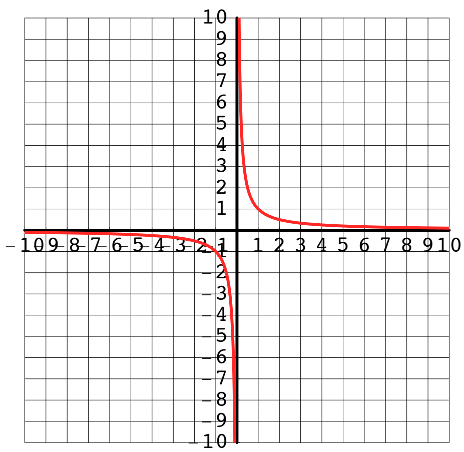
Examples:

$f(x) = \dfrac{1}{x}$	$a=0$	$b=1$	$c=1$	$d=0$
$f(x) = \dfrac{x+6}{2x-3}$	$a=1$	$b=6$	$c=2$	$d=-3$
$f(x) = \dfrac{x+1}{-3x+9}$	$a=1$	$b=1$	$c=-3$	$d=9$

The graph of $f(x) = \dfrac{ax+b}{cx+d}$

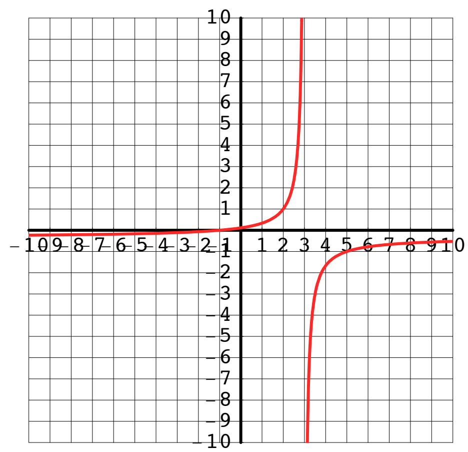
The most important features of a rational function are its asymptotes.

$$f(x) = \frac{1}{x}$$



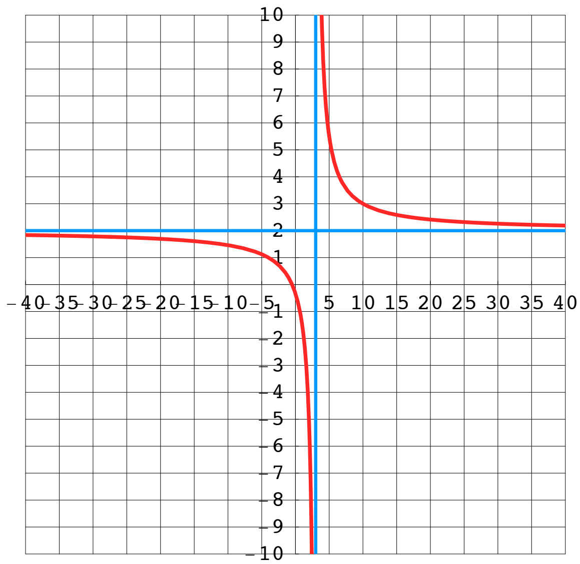
$x = 0, \; y = 0$

$$\frac{x+1}{-3x+9}$$



$x = 3, \; y = -1/3$

The graph of $f(x) = \dfrac{4x+1}{2x-6}$



As $x \to \pm\infty$, $f(x) \to 2$

$$\lim_{x \to \pm\infty} f(x) = 2$$

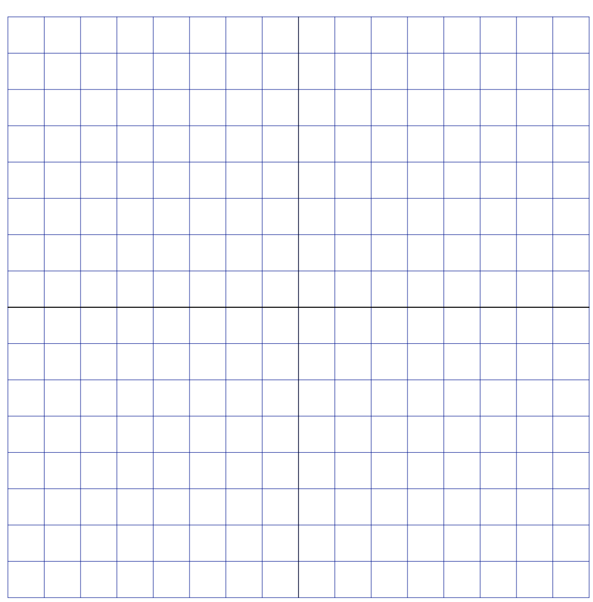
The graph of $f(x)$ has a horizontal asymptote at $y = 2$.

As $x \to 3$, $f(x) \to \pm\infty$

The graph of $f(x)$ has a vertical asymptote at $x = 3$.

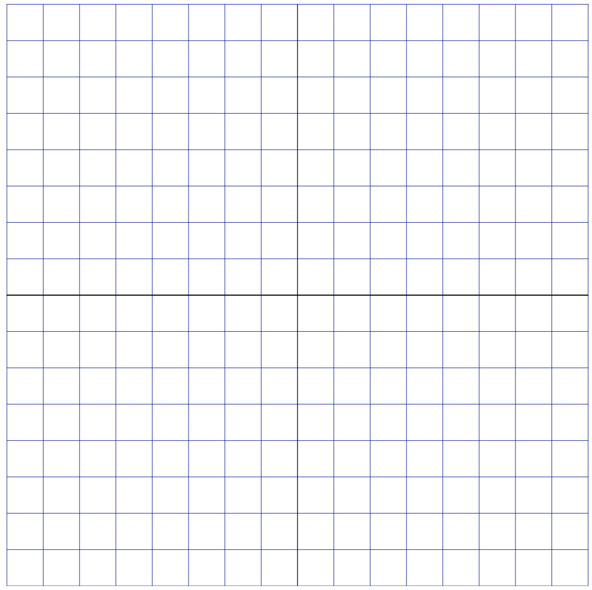
Example:

$$f(x) = \frac{x-5}{2x-7}$$



Example:

$f(x) = \dfrac{3x - 7}{x - 2}$



Application: average cost

The cost to produce a gardening book is \$8000 plus \$10.00 per copy. So the cost function is

$$C(x) = 8000 + 10x,$$

where x is the number of books produced.

What is the average cost per book, if $x = 2000$ copies are produced?

$C(2000) = 28000$ So it costs \$28,000 to produce 2000 books. That's an average of

$$\frac{\$28000}{2000} = \$14 \text{ per book.}$$

The average cost to produce x books is

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{8000 + 10x}{x},$$

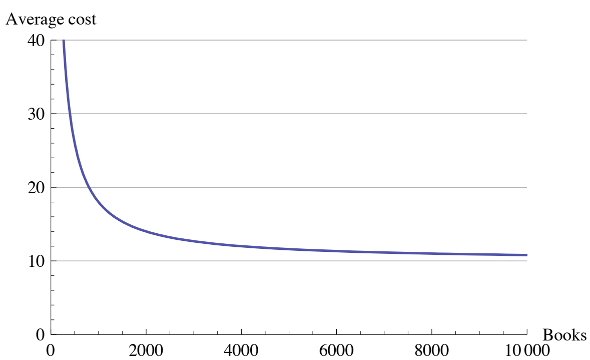
which is a rational function.

Application: average cost

$$\bar{C}(x) = \frac{8000 + 10x}{x}.$$

Find $\lim_{x \to \infty} \bar{C}(x) =$

What does this limit mean in the context of this example?



Polynomials

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

$a_n \neq 0$ is the *leading coefficient* and n is the *degree* of the polynomial.

$f(x) = 6x^2 + 6x - 1$, leading coefficient: ___, degree: ___

$f(x) = x^4 - 6x^2$, leading coefficient: ___, degree: ___

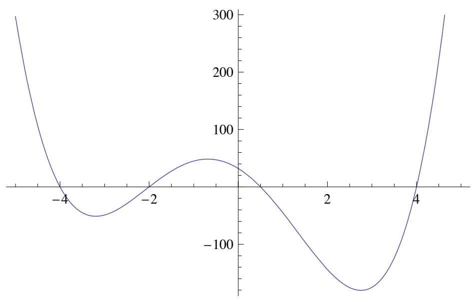
$f(x) = -2x^5 - 5x^3 + 4x + 1$, leading coefficient: ___, degree: ___

The graph of a polynomial

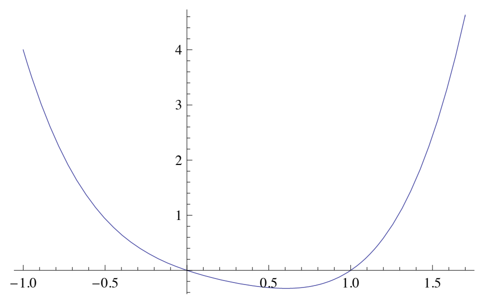
The degree n and leading coefficient a_n give us information about the graph.

How many x-intercepts?	Between 0 and n
How many turning points?	Between 0 and $n-1$

Mark the turning points and the x-intercepts on these graphs.



$f(x) = 2x^4 + 3x^3 - 34x^2 - 48x + 32$



$f(x) = x^4 - x^3 + x^2 - x$

Roots of a polynomial

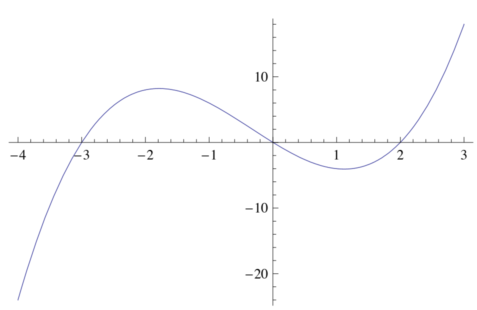
If $f(x)$ is a polynomial and a is a number such that $f(a) = 0$, then $(a, 0)$ is an x-intercept of the graph of the polynomial. Also, a is called a *root* or a *zero* of the polynomial.

The roots of a polynomial are often very hard to find, but if you can factor the polynomial, then it's easy to find its roots.

Example: $f(x) = x^3 + x^2 - 6x$. This polynomial factors as

$$x^3 + x^2 - 6x = x(x+3)(x-2).$$

So the roots are $x = 0, -3, 2$



Math of Finance, Part 1

Section 3.1, 3.2

- simple interest
- compound interest

Simple Interest

$$A = P(1 + rt)$$

- A: amount, or future value
- P: principal, or present value
- r: annual simple interest rate (decimal form)
- t time in years.

Example

$$A = P(1 + rt)$$

Find the total amount due on a loan of \$1200 at 8% simple interest at the end of 6 months.

$$\begin{aligned} A &= \text{unknown} \\ P &= 1200 \\ r &= .08 \\ t &= .5 \text{ (Note: 6 months is half a year.)} \end{aligned}$$

Then

$$A = 1200(1 + (.08)(.5)) = 1200(1 + .04) = 1200(1.04) = 1248$$

Summary: The total amount due (future value) on a loan of \$1200 at 8% simple interest at the end of 6 months is \$1248.

Example

$$A = P(1 + rt)$$

Find the total amount due on a loan of \$6,000 at 12% simple interest at the end of 9 months.

$$\begin{aligned} A &= \text{unknown} \\ P &= \\ r &= \\ t &= \text{ (Note: 9 months is 3/4 (0.75) of a year.)} \end{aligned}$$

Summary:

Warmup for Compound Interest

Problem: That new atomic cell phone you've always wanted is marked at a price of \$100. But today only, the phone store is giving a 30% discount. The sales tax is 8.25%.

You jump on it and ask the cashier to ring up the sale. The cashier does something strange. He adds in the sales tax on the full price of \$100 and then subtracts the discount of 30%. You expected him to take the discount off first and then add in the sales tax. Does it matter which way he computes the final price?

Warmup for Compound Interest

Sales tax rate: 8.25%, discount: 30% off.

Method I: add sales tax first, then subtract discount.

Method II: subtract discount first, then add sales tax.

Warmup for Compound Interest

Sales tax rate: 8.25%, discount: 30% off.
(Another way to look at the problem: multiply, don't add.)

Method I: add sales tax first, then subtract discount.

Method II: subtract discount first, then add sales tax.

Compound Interest: An example

Deposit \$100.00 into an account earning 5% compounded annually (each year). How much will you have after 10 years?

End of Year	\$Interest	\$Balance
	\$	\$100.00
1	\$5.00	\$105.00
2	\$5.25	\$110.25
3	\$5.51	\$115.76
4	\$5.79	\$121.55
5	\$6.08	\$127.63
6	\$6.38	\$134.01
7	\$6.70	\$140.71
8	\$7.04	\$147.75
9	\$7.39	\$155.13
10	\$7.76	\$162.89

Compound Interest: An example

Deposit \$100.00 into an account earning 5% compounded annually (each year). How much will you have after 10 years?

$$A = 100(1.05)^{10} = 162.89$$

Summary: You will have \$162.89 after 10 years.

Compound Interest

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

- A: amount, or future value
- P: principal, or present value
- r: annual nominal rate
- m: number of compounding periods per year
- t: time in years

Compound Interest

Example: Invest \$1200 (principal, present value) at an annual rate of 6% compounded semi-annually. How much will you have (future value) at the end of 10 years?

A	amount, or future value
$P = 1200$	principal, or present value
$r = .06$	annual nominal rate
$t = 10$	time in years
$m = 2$	number of compounding periods per year
$r/m = .06/2 = .03$	rate per compounding period
$mt = 10(2) = 20$	number of compounding periods.

$$A = 1200(1 + .03)^{20} = 1200(1.03)^{20} = 2167.33$$

Summary: You will have \$2167.33 after ten years.

Compound Interest

Example: Invest \$45,000 (principal, present value) at an annual rate of 8% compounded quarterly. How much will you have (future value) at the end of 5 years?

A (unknown)	amount, or future value
known:	
$P =$	principal, or present value
$r =$	annual nominal rate
$m =$	number of compounding periods per year
$r/m =$	rate per compounding period
$mt =$	number of compounding periods.

$A =$

Summary:

Math of Finance, Part 2

Section 3.1, 3.2

- simple interest
- compound interest
- continuous compounded interest

Three ways to compute future value

- Simple interest: $A = P(1 + rt)$
- Compound interest: $A = P\left(1 + \frac{r}{m}\right)^{mt}$
- Continuous compounded interest: $A = Pe^{rt}$

Continuous compounded interest

Example: You deposit $1000 into an account earning 12% for 10 years. The table shows the future value for various compounding periods:

	periods per year	future value	= $
1	yearly	$1000(1.12)^{10}$	= $3105.85
2	semiannually	$1000(1.06)^{20}$	= $3207.14
4	quarterly	$1000(1.03)^{40}$	= $3262.04
12	monthly	$1000(1.01)^{120}$	= $3300.39
365	daily	$1000(1+.12/365)^{3650}$	= $3319.46

Is there any limit to how large the future value can be?
Yes, the largest future value from more frequent compounding is $3320.12.

Continuous compounded interest

Formula for continuous compounded interest:

$$A = Pe^{rt}$$

r is the annual rate
t is the number of years.

For a rate of 12% for 10 years, the formula gives

$$1000e^{.12\times 10} = 1000e^{1.20} = 3320.12.$$

$$A = Pe^{rt}$$

Problem: You deposit \$12,000 into an account earning 10% compounded continuously. How much will you have at the end of 25 years?

Problem: You deposit \$650 into an account earning 4% compounded continuously. How much will you have at the end of 11 years?

Comparisons

Compound interest	$A = P\left(1 + \frac{r}{m}\right)^{mt}$
Continuous compounded interest	$A = Pe^{rt}$

Problem: You deposit \$800 into an account earning 9% interest for 6 years. What is the future value if
a. interest is compounded quarterly?
b. continuously?

a.

b.

Finding present value given future value

Example: How much money do you have to invest to have \$100,000 in 10 years? The rate is 8% compounded continuously:

In this problem we use $A = Pe^{rt}$ with $r = .08$ and $t = 10$. We know $A = 100,000$ (future value) and want to know P (present value).

$$100,000 = Pe^{.08 \times 10} = Pe^{.8} = P \times 2.225541$$

So

$$P = \frac{100,000}{2.225541} = 44932.90$$

Summary: To have a future value of \$100,000 in 10 years, you must invest \$44,932.90.

Finding present value given future value

Example: How much money do you have to invest to have \$45,200 in 5 years? The rate is 6% compounded monthly:

In this problem we use $A = P\left(1 + \frac{r}{m}\right)^{mt}$ with

$r/m =$

$mt =$

$A =$ (future value)

Summary:

Finding present value given future value

Example: How much money do you have to invest to have \$5,100 in 12 years? The rate is 7.5% compounded quarterly:

In this problem we use $A = P\left(1 + \frac{r}{m}\right)^{mt}$ with

$r/m =$

$mt =$

$A =$ (future value)

Summary:

Finding present value given future value

Example: How much money do you have to invest to have \$140,000 in 15 years? The rate is 6% compounded continuously:

In this problem we use $A = Pe^{rt}$ with

$r =$

$t =$

$A =$ (future value)

Summary:

More comparisions

Problem: You are going to invest some money for two years. Bank A offers to give 5% interest for the first year and 15% interest for the second year. Bank B offers to give 10% interest compounded yearly.

Which is better?

More comparisions

Problem: You are going to invest some money for one year. Bank A offers to give 6.2% interest compounded annually. Bank B offers to give 6.1% interest compounded monthly.

Which is better?

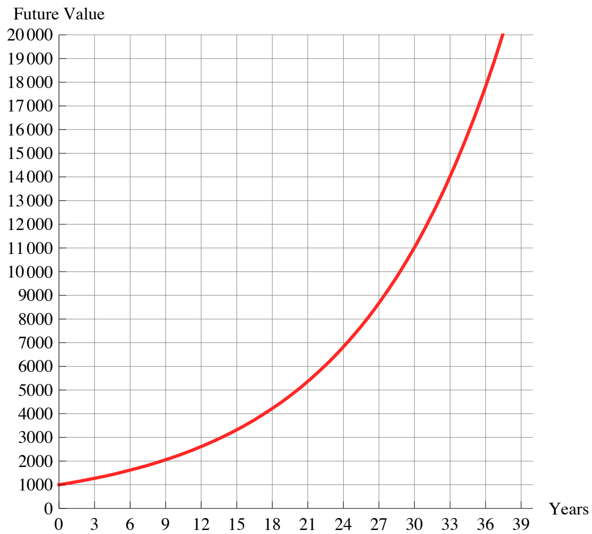
The exponential curve

You deposit \$1,000 earning 8% interest compounded continuously for t years. So the amount $A(t)$ you have is a function of the time t the money earns interest.

$$A(t) = 1000e^{.08t} = e^{.08t}$$

How long does it take for your account to grow to \$15,000?

How long does it take for your account to double in value?



Math of Finance, Part 3: solving for time

Section 3.1, 3.2

- Graphical method

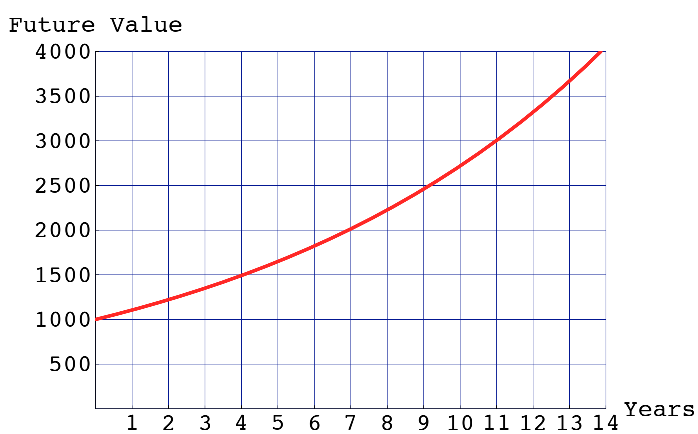
- Using logarithms

Solving the future/present value formula for time t

Example: A present value of \$1000 is invested at 10% compounded continuously. How many years are required for a future value of \$3000?

Use the graph to solve the equation for the number of years t:

$$3000 = 1000e^{(.10)t}$$



Solving the future/present value formula for time t

Use logarithms to solve the equation for the number of years t:

$$3000 = 1000e^{(.10)t}$$

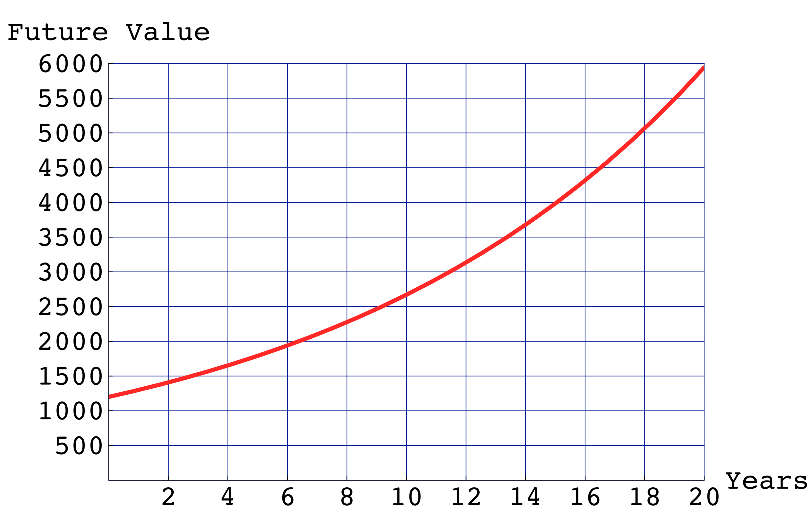
$$\begin{aligned} e^{(.10)t} &= 3 \\ \ln\left(e^{(.10)t}\right) &= \ln(3) \\ (.10)t &= \ln(3) \\ t &= \ln(3)/.10 \\ &= 10.98612 \end{aligned}$$

Summary: It takes 10.98612 years for a present value of \$1000 to grow to a future value of \$3000 at a rate of 10% compounded continuously

Solving the future/present value formula for time t

Use the graph to solve the equation for the number of years t:

$$5000 = 1200e^{(.08)t}$$



Solving the future/present value formula for time t

Use logarithms to solve the equation for the number of years t:

$5000 = 1200e^{(.08)t}$

$3600 = 1000e^{(.05)t}$

Two facts about the natural logarithm, ln

$$\ln\left(e^{x}\right) = x \tag{2}$$

$$\ln\left(a^{x}\right) = x\ln(a) \tag{3}$$

Fact (2):		Fact (3):	
$\ln\left(e^{(.03)t}\right)$	$= (.03)t$	$\ln\left((1.02)^{4t}\right)$	$= 4t\ln(1.02)$ $= 4t(0.0198026)$
$\ln\left(e^{(.09)t}\right)$	$= (.09)t$	$\ln\left((1.10)^{2t}\right)$	$= 2t\ln(1.10)$ $= 2t(0.0953102)$
$\ln\left(e^{(.06)t}\right)$	$=$	$\ln\left((1.045)^{12t}\right)$	$= 12t\ln(1.045)$ $= 12t(0.0440169)$
$\ln\left(e^{(.10)t}\right)$	$= (.10)t$	$\ln\left((1.01)^{t}\right)$	$= t\ln(1.01)$ $= t(0.0099503)$

How to use Fact (3):

Compound Interest Formula: $A = P\left(1 + \frac{r}{m}\right)^{mt}$

Problem: Deposit \$100 into an account earning 4.5% interest compounded annually. How many years will it take to have a future value of \$200? Solve for t:

$$200 = 100(1 + .045)^t.$$

$$\begin{aligned} (1.045)^t &= 2 \\ \ln\left((1.045)^t\right) &= \ln(2) \\ t\ln(1.045) &= \ln(2) \\ t &= \ln(2)/\ln(1.045) \\ t = &= 15.747302 \end{aligned}$$

Summary: It takes 15.75 years to have a future value of \$200 if a present value of \$100 earns 4.5% interest compounded annually.

How to use Fact (3)

Compound Interest Formula: $A = P\left(1 + \frac{r}{m}\right)^{mt}$

Problem

Deposit \$100 into an account earning 9% interest compounded semiannually. How many years will it take to have a future value of \$200? Solve for t:

$$200 = 100(1 + .045)^{2t}.$$

This is almost the same equation as in the previous slide. But this time we have $2t = 15.747302$. So $t = 15.747302/2 = 7.873651$ years.

Summary: It takes 7.87 years to have a future value of \$200 if a present value of \$100 earns 9% interest compounded semiannually.

Example

The present value is \$200. Interest is compounded quarterly at a rate of 10%. How many years does it take for a future value of \$500?

Compound interest formula: $A = P\left(1 + \frac{r}{m}\right)^{mt}$
$r/m = 0.10/4 = 0.025$ Solve for t:

$$
\begin{aligned}
200(1.025)^{4t} &= 500 \\
(1.025)^{4t} &= 2.5 \\
\ln\left((1.025)^{4t}\right) &= \ln(2.5) \\
4t \ln(1.025) &= \ln(2.5) \\
t &= \ln(2.5)/(4\ln(1.025)) \\
t &= 9.277
\end{aligned}
$$

Summary: It takes 9.277 years to have a future value of \$500 if a present value of \$200 earns 10% compounded quarterly.

Problem

The present value is \$1200. Interest is compounded monthly at a rate of 8%. How many years does it take for a future value of \$2000?

Problem

The present value is \$1800. Interest is compounded quarterly at a rate of 12%. How many years does it take for a future value of \$3200?

Problem

The present value is \$440. Interest is compounded semi-annually at a rate of 7%. The number of years, t, it takes for a future value of \$900 is given by the expression

$$t = \frac{\ln(a)}{b\ln(c)}.$$

Find a, b, c.