Trace-minimal graphs and D-optimal weighing designs

Bernardo M. Ábrego
Silvia Fernández-Merchant
Michael G. Neubauer
William Watkins
California State University, Northridge

March 23, 2004, v.115

Abstract

Let $\mathcal{G}(v,\delta)$ be the set of all δ-regular graphs on v vertices. Certain graphs from among those in $\mathcal{G}(v,\delta)$ with maximum girth have a special property called trace-minimality. In particular, all strongly regular graphs with no triangles and some cages are trace-minimal. These graphs play an important role in the statistical theory of D-optimal weighing designs.

Each weighing design can be associated with a $(0,1)$-matrix. Let $M_{m,n}(0,1)$ denote the set of all $m \times n$ (0,1)-matrices and let

$$G(m,n) = \max\{\det X^T X : X \in M_{m,n}(0,1)\}.$$

A matrix $X \in M_{m,n}(0,1)$ is a D-optimal design matrix if $\det X^T X = G(m,n)$. In this paper we exhibit some new formulas for $G(m,n)$ where $n \equiv -1 \pmod 4$ and m is sufficiently large. These formulas depend on the congruence class of $m \pmod n$. More precisely, let $m = nt + r$ where $0 \leq r < n$. For each pair n, r, there is a polynomial $P(n,r,t)$ of degree n in t, which depends only on n, r, such that $G(nt + r, n) = P(n,r,t)$ for all sufficiently large t. The polynomial $P(n,r,t)$ is computed from the characteristic polynomial of the adjacency matrix of a trace-regular graph whose degree of regularity and number of vertices depend only on n and r. We obtain explicit expressions for the polynomial $P(n,r,t)$ for many pairs n, r. In particular we obtain formulas for $G(nt + r, n)$ for $n = 19, 23$, and 27, all $0 \leq r < n$, and all sufficiently large t. And we obtain families of formulas for $P(n,r,t)$ from families of trace-minimal graphs including bipartite graphs obtained from finite projective planes, generalized quadrilaterals, and generalized hexagons.

Keywords: D-optimal weighing design, trace-minimal graph, regular graph, strongly regular graph, girth, cages, generalized polygons
AMS Subject Classification:
Primary: 05C50, 62K05
Secondary: 15A36, 05B25, 15A15, 05B20

1 Introduction

In [AFNW], the present authors established a relationship between certain regular graphs and D-optimal designs for weighing $n \equiv -1 \pmod 4$ objects. We now further develop the graph-theoretic concept of trace-minimality and use it to obtain additional results on D-optimality.

1.1 Trace-minimal graphs

Let $\mathcal{G}(v, \delta)$ be the set of all δ-regular graphs on v vertices. We call $\mathcal{G}(v, \delta)$ a *graph class*. Let $A(G)$ be the adjacency matrix of a graph G. The characteristic polynomial of $A(G)$ is denoted by $\mathrm{ch}(G, x)$ and the spectrum of $A(G)$ is denoted by $\mathrm{spec}(A(G))$. We also refer to $\mathrm{ch}(G, x)$ as the characteristic polynomial of the graph G and $\mathrm{spec}(A(G))$ as the spectrum of G. Since $A(G)$ is a symmetric $(0,1)$-matrix with zeros on the diagonal and δ ones in each row, $\mathrm{tr}A(G) = 0$ and $\mathrm{tr}A(G)^2 = \delta v$. These traces do not depend on the structure of the graph G. However, for $i \geq 3$, $\mathrm{tr}A(G)^i$ does depend on the structure of the graph. Indeed the (j, j) entry of $A(G)^i$ equals the number of closed walks of length i that start and end at vertex j. For $G \in \mathcal{G}(v, \delta)$ define the *trace sequence* of G by $\mathrm{TR}(G) = (\mathrm{tr}A(G)^3, \mathrm{tr}A(G)^4, \ldots, \mathrm{tr}A(G)^n)$.

The trace sequence induces an order relation on the graphs in $\mathcal{G}(v, \delta)$. Let $G, H \in \mathcal{G}(v, \delta)$. We say G is *trace-dominated* by H if $\mathrm{TR}(G)$ is less than or equal to $\mathrm{TR}(H)$ in lexicographic order. In other words, G is trace-dominated by H if either $\mathrm{tr}A(G)^i = \mathrm{tr}A(H)^i$ for all i (in which case $\mathrm{spec}(A(G)) = \mathrm{spec}(A(H))$) or there exists a positive integer $3 \leq k \leq n$ such that $\mathrm{tr}A(G)^i = \mathrm{tr}A(H)^i$, for $i < k$ and $\mathrm{tr}A(G)^k < \mathrm{tr}A(H)^k$. If G is trace-dominated by all graphs in $\mathcal{G}(v, \delta)$, then we say that G is *trace-minimal* in $\mathcal{G}(v, \delta)$. Since $\mathcal{G}(v, \delta)$ is finite, there always exist trace-minimal graphs in $\mathcal{G}(\delta, v)$ and clearly they all have the same characteristic polynomial. Some graph classes $\mathcal{G}(v, \delta)$, contain non-isomorphic graphs, each of which is trace-minimal. However, the smallest example known to us are two nonisomorphic cages in the graph class $\mathcal{G}(70, 3)$. (See Sections 4.5 and 4.4.1.)

In addition to their application to the theory of D-optimal weighing designs, trace-minimal graphs are of independent interest. Indeed many well-known classes of regular graphs are trace-minimal including strongly regular graphs with no triangles, some cages, and the incidence graphs for various finite geometries. These and other families of trace-minimal graphs are given in Section 4. And it is clear from the definition that a trace-minimal graph $G \in \mathcal{G}(v, \delta)$ must have maximum girth g among all graphs in its graph class $\mathcal{G}(v, \delta)$. Furthermore, G must have the fewest number of g-cycles of any graph in the same class. We will describe more fully the connection between trace-minimality and girth in Section 2.

1.2 D-optimal weighing designs

Let $M_{m,n}(0, 1)$ be the set of all $m \times n$ matrices all of whose entries are either 0 or 1. Removed from its statistical setting, our problem is to determine, for each pair of positive integers m, n, the maximum value of the determinant of the $n \times n$ matrix $X^T X$ for $X \in M_{m,n}(0, 1)$. If $m < n$ then $\det X^T X = 0$, so we will assume throughout that $m \geq n$. Let

$$G(m, n) = \max\{\det X^T X : X \in M_{m,n}(0, 1)\}.$$

A matrix $X \in M_{m,n}(0, 1)$ is *D-optimal* if $\det X^T X = G(m, n)$.

Statistical weighing designs date back to 1935 [Ya] and the 1940s [Ho] [Mo]. The goal is to estimate the weights of n objects using a single-pan (spring) scale. (We do not assume that the scale is accurate, its errors have a distribution.) Several objects are placed on the scale at once and their total weight is noted. The information about which objects are place on the scale is encoded as a $(0,1)$-n-tuple whose jth coordinate is 1 if object j is included in the weighing and 0 if not. (The weights of n objects cannot be reasonably estimated in fewer than n weighings, so the restriction $m \geq n$ makes statistical sense.) With m weighings the corresponding $(0,1)$-n-tuples form the rows of an $m \times n$ *design matrix* $X \in M_{m,n}(0,1)$. Certain design matrices give better estimates of the weights of the n objects than others. For example, under certain assumptions about the distribution of errors of the scale, D-optimal design matrices give confidence regions for the n-tuple of weights of the objects that have minimum volume. There are other standards for evaluating the efficiency of a design matrix such as A-optimality which corresponds to a design matrix X for which $\mathrm{tr}(X^T X)^{-1}$ is smallest. See [Pu] for an overview.

The problem also arises in a geometric settings. If $X \in M_{m,n}(0,1)$, the columns of X are vertices on the unit cube in $\mathbb{R}^m$. The simplex spanned by the origin and each of the n columns of X has an n-dimensional volume equal to $(1/n!)\sqrt{\det X^T X}$. So the problem of finding $G(m,n)$ is equivalent to finding the volume of the largest n-simplex on the vertices of the unit cube in $\mathbb{R}^m$. See [HKL] for an up-to-date discussion and extensive list of references.

In general, the value of $G(m,n)$ is not known. Although there are results for some pairs m,n, the only values of n for which $G(m,n)$ is known for all $m \geq n$ are $n = 1,2,3,4,5,6$. See [HKL] for $n = 2,3$, [NWZ2] for $n = 4,5$, and [NWZ3] for $n = 6$. Prior to this paper, the only values of n for which $G(m,n)$ was known for all but a finite number of values of m (that is, for m sufficiently large) were $n = 7,11,15$. See [NW] for $n = 7$, and [AFNW] for $n = 11,15$.

For example, the following formula for $n = 7$ was conjectured in [HKL]:

$$G(7t + r, 7) = 2^{10}(t + 1)^r t^{7-r}. \tag{1}$$

In [NW], the formula was shown to be true for all $t \geq 15$ and $0 \leq r \leq 6$. In general, if $n \equiv -1 \pmod 4$ and $m = nt + r$ where $0 \leq r < n$, then $G(m,n)$ is a polynomial $P(n,r,t)$ in t of degree n which depends only on n and r. Indeed the authors of [AFNW] have shown that such a polynomial always exists. To be precise, we state the following theorem:

Theorem 1.1 *[AFNW] For each* $n \equiv -1 \pmod 4$ *and each* $0 \leq r < n$*, there exists a polynomial* $P(n,r,t)$ *in* t *of degree* n *such that*

$$G(nt + r, n) = P(n,r,t), \tag{2}$$

for all sufficiently large t*.*

Thus for each pair n,r, we define the polynomial $P(n,r,t)$ to be the one for which Equation (2) holds for all sufficiently large t. In some cases, this polynomial can be computed explicitly as in Equation (1). An explicit expression for the polynomial $P(n,r,t)$ can be obtained from a trace-minimal graph from a certain graph class $\mathcal{G}(v,\delta)$, where v and δ depend only on n,r. Thus in principle $P(n,r,t)$ can be obtained by comparing the trace sequences of all graphs in a graph class. The trace-minimal graphs for $n = 7$ and $0 \leq r < 7$ are rather easy to find and all polynomials $P(n,r,t)$ and their associated trace-minimal graphs for $n = 7,11,15$ and all $0 \leq r < n$ are exhibited in [AFNW].

In this paper we use trace-minimal graphs to obtain expressions for the polynomials $P(n,r,t)$ for many new pairs (n,r). In particular, we exhibit $P(n,r,t)$ for $n = 19,23$, and 27 and all $0 \leq r < n$. We present these and other new polynomials $P(n,r,t)$ in Section 5. In Section 1.3 we describe the relationship between trace-minimal graphs and the polynomials $P(n,r,t)$. In Section 4 we exhibit several families of trace-minimal graphs, some of which are associated with finite projective planes and other combinatorial constructs. These graphs correspond to formulas for $P(n,r,t)$ for infinite families of pairs (n,r) including families where r is large and small compared with n, and where r is near $n/2$.

Before going on, we should say that for some pairs n,r, where $n \equiv -1 \pmod 4$, anomalies occur for small values of m. For example, let $n = 7$ and $m = 18$ so that $t = 2$ and $r = 4$. From Equation (1), $G(7t+4, 7) = P(7,4,t)$ for all sufficiently large t, where $P(7,4,t) = 4\ 2^8(t+1)^4t^3$. Apparently $t = 2$ is not large enough. It is not hard to obtain an 18×7 design matrix A such that $\det A^TA > P(7,4,2) = 663,552$. A simple computer program produced a design matrix such that $\det A^TA = 684,375$. Thus $G(18,7)$ is at least 684,375. We suspect that there are anomalies for all $n \geq 7$, but the matter has not been investigated.

1.3 Trace-minimal graphs give expressions for $P(n,r,t)$

In this section we summarize the results in [AFNW] relating the polynomial $P(n,r,t)$ to a trace-minimal graph. Let $n \equiv -1 \pmod 4$ so that $n = 4p-1$ for some positive integer p. Let $m = nt + r$, where the remainder r satisfies $0 \leq r < n$. The formulas for $P(n,r,t)$ from [AFNW] depend on the congruence class of $r \pmod 4$. We begin with the cases $r \equiv 1, 2 \pmod 4$:

Theorem 1.2 *Let $r = 4d+1$. Let G be a trace-minimal graph in $\mathcal{G}(2p,d)$. Then*

$$P(n,r,t) = \frac{4(t+1)[\mathrm{ch}(G,pt+d)]^2}{t^2}. \tag{3}$$

Theorem 1.3 *Let $r = 4d+2$. Let G be a trace-minimal graph in $\mathcal{G}(2p,p+d)$. Then*

$$P(n,r,t) = \frac{4t[\mathrm{ch}(G,pt+d)]^2}{(t-1)^2}. \tag{4}$$

To state the results for $r \equiv -1, 0 \pmod 4$, we need to define a notion analogous to trace-minimality for bipartite graphs. Let $\mathcal{B}(2v,\delta)$ be the set of all δ-regular bipartite graphs on $2v$ vertices and let $B \in \mathcal{B}(2v,\delta)$. It follows from the regularity of B that each of the sets of vertices in the bipartition has cardinality v. Without loss of generality, we may assume that the sets of vertices in the bipartition are $\{1,2,\ldots,v\}$ and $\{v+1, v+2, \ldots, 2v\}$. Thus the adjacency matrix of B is of the form

$$A(B) = \begin{bmatrix} 0 & N(B) \\ N(B)^T & 0 \end{bmatrix},$$

where $N(B)$ is a $v \times v$ $(0,1)$-matrix having exactly δ ones in each row and each column.

It is clear that $\mathrm{tr}A(B)^i = 0$ if i is odd and that $\mathrm{tr}A(B)^{2j} = 2\mathrm{tr}((N(B)^TN(B))^j)$ otherwise. For $j = 1$, $\mathrm{tr}((N(B)^TN(B)) = v\delta$, for all $B \in \mathcal{B}(2v,\delta)$.

A graph $B \in \mathcal{B}(2v,\delta)$ is *bipartite-trace-minimal* in $\mathcal{B}(2v,\delta)$ if B is trace-dominated by all graphs in $\mathcal{B}(2v,\delta)$. Clearly if G is trace-minimal in $\mathcal{G}(2v,\delta)$ and bipartite, then it is bipartite-trace-minimal. But some bipartite-trace-minimal graphs are not trace-minimal.

Theorem 1.4 *Let $r = 4d-1$. Suppose $p/2 \leq d < p$. Let G be a trace-minimal graph in $\mathcal{G}(4p, 3p+d-1)$. Then*

$$P(n,r,t) = \frac{4\mathrm{ch}(G, pt+d-1)}{t-3}. \tag{5}$$

Suppose $0 \leq d < p/2$. Let B be a bipartite-trace-minimal graph in $\mathcal{B}(4p, d)$. Then

$$P(n,r,t) = \frac{4(p(t-1)+2d)\mathrm{ch}(B, pt+d)}{t(pt+2d)}. \tag{6}$$

Theorem 1.5 *Let $r = 4d$. Suppose $0 \leq d \leq p/2$. Let G be a trace-minimal graph in $\mathcal{G}(4p, d)$. Then*

$$P(n,r,t) = \frac{4\mathrm{ch}(G, pt+d)}{t}. \tag{7}$$

Suppose $p/2 < d < p$. Let B be a bipartite-trace-minimal graph on in $\mathcal{B}(4p, p+d)$. Then

$$P(n,r,t) = \frac{4(pt+2d)\mathrm{ch}(B, pt+d)}{(t-1)(p(t+1)+2d)}. \tag{8}$$

Equipped with these four theorems, one can translate the problem of finding an explicit expression of $P(n,r,t)$ for a given $n \equiv -1 \pmod 4$ and remainder $0 \leq r < n$ into the problem of finding an appropriate trace-minimal or bipartite-trace-minimal graph. For example suppose $n = 19$ and $r = 13$ so that $p = 5$ and $r = 4d+1$, where $d = 3$. This case falls within the scope of Theorem 1.2 and we seek a trace-minimal graph in $\mathcal{G}(10, 3)$. The Petersen graph G, which is 3-regular on 10 vertices, is trace-minimal (see Section 4.3). Since $\mathrm{ch}(G,x) = (x-3)(x-1)^5(x+2)^4$, Theorem 1.2 gives

$$\begin{aligned} P(19,13,t) &= \frac{4(t+1)[\mathrm{ch}(G, 5t+3)]^2}{t^2} \\ &= 20(5t+2)^{10}(5t+5)^9, \end{aligned}$$

which proves one of the formulas in Theorem 5.21.

In a similar manner, we prove all of the theorems in Section 5 by exhibiting the required trace-minimal and bipartite-trace-minimal graphs and the corresponding characteristic polynomials. The proofs that each graph is trace-minimal or bipartite-trace-minimal will be given later in Section 4.

2 Sufficient conditions for trace-minimality

All trace-minimal graphs $G \in \mathcal{G}(v, \delta)$ have maximum girth in their graph class $\mathcal{G}(v, \delta)$. And many of the graphs we catalog in Section 4 satisfy one of two conditions involving girth that are sufficient for trace-minimality. Thus it is convenient to state these conditions before listing these families of graphs.

Let $\mathrm{cyc}(G, i)$ denote the number of cycles of length i in the graph G. This first condition for trace-minimality is the following:

Theorem 2.1 *Let G be a graph with maximum girth g in $\mathcal{G}(v, \delta)$. Suppose that for every graph $H \in \mathcal{G}(v, \delta)$, there exists an integer $k \leq 2g-1$ such that $\mathrm{cyc}(G, q) = \mathrm{cyc}(H, q)$ for $q < k$ and $\mathrm{cyc}(G, k) < \mathrm{cyc}(H, k)$. Then G is trace-minimal in $\mathcal{G}(v, \delta)$.*

In particular, if G is the only graph in $\mathcal{G}(v,\delta)$ with maximum girth g, then $\mathrm{cyc}(G,q) < \mathrm{cyc}(H,q)$ for all $q \le g$ and all $H \in \mathcal{G}(v,\delta)$. Thus we have the following corollary:

Corollary 2.2 *Let G be the only graph in $\mathcal{G}(v,\delta)$ with maximum girth. Then G is the only trace-minimal graph in $\mathcal{G}(v,\delta)$.*

The next condition involves the number of distinct eigenvalues in the spectrum (of the adjacency matrix) of G. Suppose a graph G has girth g and its adjacency matrix $A(G)$ has $k+1$ distinct eigenvalues. Then [CDS, p88], the diameter D of G satisfies $D \le k$. It is clear that $\lfloor g/2 \rfloor \le D$. Thus $g \le 2k$ if the girth g is even and $g \le 2k+1$ if g is odd. We analyze the case of equality in the next theorem.

Theorem 2.3 *Let G be a connected regular graph with girth g and suppose that $A(G)$ has $k+1$ distinct eigenvalues. If g is even then $g \le 2k$ with equality only if G is trace-minimal. If g is odd then $g \le 2k+1$ with equality only if G is trace-minimal.*

The proofs of these Theorems are in Section 7.

3 Graph definitions and notation

We begin with a list of some common graph notation that is used in this paper:

I_v	the graph consisting of v independent vertices (no edges)
K_v	the complete graph on v vertices
$K_{v,v}$	the complete bipartite graph with v vertices in each of the bipartition sets
C_v	the cycle with v vertices
vK_2	a matching of v edges on $2v$ vertices
$K_{2v} - vK_2$	the complete graph on $2v$ vertices with a matching of v edges removed
$K_{v,v} - vK_2$	the complete bipartite graph with a matching of v edges removed
G^{comp}	the complement of a graph G
B^{bcomp}	the bipartite-complement of a bipartite graph B (See 3.1.)
$G + H$	the direct sum of graphs G and H
kG	the direct sum of k copies of G
$G \bigtriangledown H$	the join of graphs G and H (See 3.3.)
$G^{(l)}$	the join of l copies of the graph G

3.1 Bipartite complement

Let $B \in \mathcal{B}(2v,\delta)$ with adjacency matrix

$$A(B) = \begin{bmatrix} 0 & N(B) \\ N(B)^T & 0 \end{bmatrix}.$$

The *bipartite complement* of B is the graph $B^{\mathrm{bcomp}} \in \mathcal{B}(2v, v-\delta)$ with adjacency matrix given by

$$A(B^{\mathrm{bcomp}}) = \begin{bmatrix} 0 & J - N(B) \\ J - N(B)^T & 0 \end{bmatrix},$$

where J is the matrix all of whose entries are one. It is easy to see that $\mathrm{tr}A(B)^j = \mathrm{tr}A(B^{\mathrm{bcomp}})^j = 0$ if j is odd and that

$$\frac{\mathrm{ch}(B^{\mathrm{bcomp}}, x)}{x^2 - (v-\delta)^2} = \frac{\mathrm{ch}(B, x)}{x^2 - \delta^2}. \tag{9}$$

So apart from the eigenvalues $\pm\delta$ of $A(B)$ and $\pm(v-\delta)$ of B^{bcomp}, the spectra of B and B^{bcomp} are the same. Thus

$$\mathrm{tr}A(B^{\mathrm{bcomp}})^{2j} - 2(v-\delta)^{2j} = \mathrm{tr}A(B)^{2j} - 2\delta^{2j}.$$

The next Lemma follows easily from that fact.

Lemma 3.1 *If B is bipartite-trace-minimal in $\mathcal{B}(2v, \delta)$, then B^{bcomp} is bipartite-trace-minimal in $\mathcal{B}(2v, v-\delta)$.*

3.2 Complement

The relationship between the characteristic polynomials of a graph $G \in \mathcal{G}(v, \delta)$ and its complement $G^{\mathrm{comp}} \in \mathcal{G}(v, \delta')$, where $\delta+\delta' = v-1$, follows from $A(G)+A(G^{\mathrm{comp}}) = J-I$. Since we will need to compute $\mathrm{ch}(G, x)$ from $\mathrm{ch}(G^{\mathrm{comp}}, x)$, suppose that $\mathrm{ch}(G^{\mathrm{comp}}, x) = (x-\delta')p(x)$ Then $\mathrm{ch}(G, x) = \pm(x-\delta)p(-x-1)$, where the choice of plus-minus is made so that the polynomial is monic. More succinctly, the relationship is:

$$\frac{\mathrm{ch}(G, x)}{x - \delta} = \pm\frac{\mathrm{ch}(G^{\mathrm{comp}}, -x-1)}{x + v - \delta}. \tag{10}$$

3.3 Join

The *join* of two graphs $G_i \in \mathcal{G}(v_i, \delta_i)$, $i = 1, 2$, is the graph on $v = v_1 + v_2$ vertices defined by $G \bigtriangledown H = (G^{\mathrm{comp}} + H^{\mathrm{comp}})^{\mathrm{comp}}$. Then $A(G \bigtriangledown H) = \begin{bmatrix} A(G) & J \\ J & A(H) \end{bmatrix}$. The characteristic polynomial of $A(G_1 \bigtriangledown G_2)$ is related to the characteristic polynomials of $A(G_1)$ and $A(G_2)$ in the following way [CDS, FG]:

$$\frac{\mathrm{ch}(G_1 \bigtriangledown G_2, x)}{(x-\delta_1)(x-\delta_2) - v_1 v_2} = \frac{\mathrm{ch}(G_1, x)\mathrm{ch}(G_2, x)}{(x-\delta_1)(x-\delta_2)}.$$

The join of two regular graphs is not regular unless $v_2 + \delta_1 = v_1 + \delta_2$ and in this case relationship of the characteristic polynomials is given by

$$\frac{\mathrm{ch}(G_1 \bigtriangledown G_2, x)}{(x-\delta)(x+v-\delta))} = \frac{\mathrm{ch}(G_1, x)\mathrm{ch}(G_2, x)}{(x-\delta_1)(x-\delta_2)}, \tag{11}$$

where $\delta = v_2 + \delta_1 = v_1 + \delta_2$. Furthermore the following Lemma holds:

Lemma 3.2 *Let $G_1 \in \mathcal{G}(v_1, \delta_1)$ and $G_2 \in \mathcal{G}(v_2, \delta_2)$. If $v_2+\delta_1 = v_1+\delta_2 = \delta$, then $G_1 \bigtriangledown G_2 \in \mathcal{G}(v_1+v_2, \delta)$. If $G_1 \bigtriangledown G_2$ is trace-minimal then G_1 and G_2 are trace-minimal.*

Proof: Let $v = v_1 + v_2$. Then,

$$\mathrm{tr}A(G_1 \bigtriangledown G_2)^j - \delta^j - (\delta - v)^j = \mathrm{tr}A(G_1)^j - \delta_1^j + \mathrm{tr}A(G_2)^j - \delta_2^j.$$

It follows that if a graph H_1 in $\mathcal{G}(v_1, \delta_1)$ is trace-dominated by G_1, then $H_1 \bigtriangledown G_2$ is trace-dominated by $G_1 \bigtriangledown G_2$. So if $G_1 \bigtriangledown G_2$ is trace-minimal, then G_1 is trace-minimal and, by a similar argument, G_2 is also trace-minimal. ∎

4 Families of trace-minimal graphs

4.1 Graphs unique in their graph class

Each graph G in the next theorem is the only one in its graph class. Thus G is trace-minimal (bipartite-trace-minimal). The characteristic polynomials of the adjacency matrices for these graphs are well known.

Lemma 4.1 *The following graphs are trace-minimal (bipartite-trace-minimal) in their graph class. The characteristic polynomial is given:.*

graph class	G	$\mathrm{ch}(G, x)$
$\mathcal{G}(v, 0)$	I_v	x^v
$\mathcal{B}(2v, 0)$	I_{2v}	x^{2v}
$\mathcal{G}(v, v-1)$	K_v	$(x - (v-1))(x+1)^{v-1}$
$\mathcal{G}(2v, 1)$ *or* $\mathcal{B}(2v, 1)$	vK_2	$(x-1)^v(x+1)^v$
$\mathcal{B}(2v, v-1)$	$K_{v,v} - vK_2$	$(x^2 - (v-1)^2)(x^2-1)^{v-1}$
$\mathcal{B}(2v, v)$	$K_{v,v}$	$(x-v)(x+v)x^{2v-2}$
$\mathcal{G}(2v, 2v-2)$	$K_{2v} - vK_2 = I_2^{(v)}$	$(x - 2v + 2)(x+2)^{v-1}x^v$.

4.2 Cycles and related graphs

Let $\mathrm{Tch}_v(x)$ stand for the vth Tchebychev polynomial of the first kind, which is characterized by the identity $\cos(vx) = \mathrm{Tch}_v(\cos x)$. See [Ri] for details.

Lemma 4.2 *The cycle graph $G = C_v$ is the only trace-minimal graph in $\mathcal{G}(v, 2)$. Its characteristic polynomial is:*

$$\mathrm{ch}(G, x) = 2\mathrm{Tch}_v(x/2) - 2.$$

The bipartite graph $B = K_{v,v} - C_{2v} = C_{2v}^{\mathrm{bcomp}}$ is the only bipartite-trace-minimal graph in $\mathcal{B}(2v, v-2)$. Its characteristic polynomial is:

$$\mathrm{ch}(B, x) = \frac{(x^2 - (v-2)^2)}{x^2 - 4}\left(2\mathrm{Tch}_{2v}(x/2) - 2\right).$$

Proof: The cycle C_v is the only graph in $\mathcal{G}(v, 2)$ with girth v, which is the maximal. Thus by Corollary 2.2, C_v is trace-minimal.

The second part of the lemma follows from Lemma 3.1 and Equation (9). ∎

4.3 Strongly regular graphs

A graph G on N vertices is *strongly regular* on the parameters $(N, \delta, \lambda, \mu)$ if it is a δ-regular graph that satisfies the following two conditions:

(i) If u, v are vertices and (u, v) is an edge of G, then there are λ additional vertices that are joined to both u and to v by an edge.

(ii) If u, v are vertices and (u, v) is not an edge of G, then there are μ vertices that are joined to both u and to v by an edge.

Lemma 4.3 *Let G be a connected strongly δ-regular graph with no 3-cycles. Then G is trace-minimal.*

This result follows immediately from Theorem 2.3. Every strongly regular graph $G \in \mathcal{G}(v, \delta)$ has only $3 = k + 1$ distinct eigenvalues. Since there are no 3-cycles in G, the girth g of G must be at least 4. If g is odd, then $5 \leq g \leq 2k + 1 = 5$. Hence $g = 2k + 1 = 5$. If g is even, then $g = 2k = 4$.

The following table lists all seven known strongly regular graphs G with parameter $\lambda = 0$ (no triangles) along with the girth and characteristic polynomial for $A(G)$. (See [God]; see [CRC] for Higman-Sims (77).)

Name	girth	parameter set	graph class	characteristic polynomial
C_5	5	$(5, 2, 0, 1)$	$\mathcal{G}(5, 2)$	$(x-2)(x^2+x-1)^2$
Petersen	5	$(10, 3, 0, 1)$	$\mathcal{G}(10, 3)$	$(x-3)(x-1)^5(x+2)^4$
Clebsh	4	$(16, 5, 0, 2)$	$\mathcal{G}(16, 5)$	$(x-5)(x-1)^{10}(x+3)^5$
Hoffman-Singleton	5	$(50, 7, 0, 1)$	$\mathcal{G}(50, 7)$	$(x-7)(x-2)^{28}(x+3)^{21}$
Gewirtz	4	$(56, 10, 0, 2)$	$\mathcal{G}(56, 10)$	$(x-10)(x-2)^{35}(x+4)^{20}$
Higman-Sims (77)	4	$(77, 16, 0, 4)$	$\mathcal{G}(77, 16)$	$(x-16)(x+6)^{21}(x-2)^{55}$
Higman-Sims	4	$(100, 22, 0, 6)$	$\mathcal{G}(100, 22)$	$(x-22)(x-2)^{77}(x+8)^{22}$

4.4 Generalized polygons

Finite projective planes are one example of a class of geometries known as generalized polygons. (See [vM, p.5] for the definition and other details.) We are interested in generalized polygons of order q. Each line contains exactly $q + 1$ points and each point is on $q + 1$ lines. Generalized n-gons of order q with $n \geq 3$ exist if and only if $n = 3, 4, 6$ and they have been constructed whenever q is a power of a prime. Generalized 3-gons are projective planes, generalized 4-gons are called generalized quadrangles, and generalized 6-gons are called generalized hexagons.

The *incidence graph* [vM, p. 3] of a finite geometry Γ is the bipartite graph whose vertices are bipartitioned into the lines and the points with an edge whenever a point and a line are incident. Thus the adjacency matrix for the incidence graph G of a finite geometry is for the form

$$A(G) = \begin{bmatrix} 0 & N(G) \\ N(G)^T & 0 \end{bmatrix},$$

where $N(G)$ is the line-point incidence matrix of Γ.

In the next three sections, we shall show that the spectrum of G has only $n+1$ distinct eigenvalues, from which it follows by Theorem 2.3 that G is trace-minimal.

Theorem 4.4 *Let G be the incidence graph for a generalized n-gon of order q. Then G is trace-minimal.*

We need some facts about generalized n-gons.

Lemma 4.5 ([vM]) *Let Γ be a generalized n-gon of order q and let G be the incidence graph of Γ. Then*

The number of points and the number of lines in Γ is $v = (q^n - 1)/(q - 1)$.

Each line in Γ contains $q + 1$ points.

$G \in \mathcal{B}(2v, q + 1)$.

The girth of G is $2n$.

In the next three sections, we compute the characteristic polynomials of the incidence graphs for projective planes, generalized quadrangles, and generalized hexagons.

4.4.1 Projective planes

Let Γ be a finite projective plane of order q. (The parameter q is a power of a prime for all known projective planes.) There are v points and v lines, where $v = (q^3 - 1)/(q - 1)$. (See [vLW, p. 197].) Let $d = q + 1$ and $r = 4d + 1 = 4q + 5$. Let $PP(q) \in \mathcal{G}(2v, q + 1)$ be the incidence graph for Γ. Then $N(PP(q))^T N(PP(q)) = qI + J$, since each line contains $q + 1$ points and distinct lines intersect in one point. Similarly $N(PP(q))N(PP(q))^T = qI + J$. The eigenvalues of $qI + J$ are $v + q = (q + 1)^2$ and q ($v - 1$ times). Thus $\mathrm{ch}(PP(q), x) = (x^2 - (q + 1)^2)(x^2 - q)^{v-1}$ so $A(PP(q))$ has only four eigenvalues. The girth of $PP(q)$ is 6. Thus by Theorem 2.3 $PP(q)$ is trace-minimal. We have proved the following lemma:

Lemma 4.6 *Let $PP(q)$ be the incidence graph of a projective plane of order q and let $v = (q^3-1)/(q-1)$. Then $PP(q)$ is a trace-minimal graph in $\mathcal{G}(2v, q+1)$ and*

$$\mathrm{ch}(PP(q), x) = (x^2 - (q + 1)^2)(x^2 - q)^{v-1}.$$

In general, projective planes of order q are not unique, that is not isomorphic as planes. Indeed there are two non-isomorphic projective planes of order $q = 9$, both of whose incidence graphs are trace-minimal in $\mathcal{G}(182, 10)$. One of them is the plane π_F constructed from the field F with 9 elements. Another is π_H, the Hughes plane. Not only are these planes nonisomorphic (as planes), but their incidence graphs, which are in $\mathcal{G}(182, 10)$, are nonisomorphic as graphs. This follows from the fact that π_F contains a Fano configuration whereas π_H does not. Thus the graphs are not isomorphic since the Fano configuration induces a subgraph of π_H that is not present in π_F. See [St, p.59] for definitions and details.

4.4.2 Generalized quadrangles

Let Γ be a generalized quadrangle of order q, where q is a power of a prime. There are $v = (q^4-1)/(q-1)$ points and lines in Γ. Let $GQ(q)$ be the incidence graph of Γ. Then $GQ(q) \in \mathcal{G}(2v, q+1)$. Let $B = N(GQ(q))^T N(GQ(q))$, where $N(GQ(q))$ is the line-point incidence matrix for Γ. Using arguments similar to those in [vM, Appendix A], we get

$$B_{i,j} = \begin{cases} q+1, \text{ if } i=j \\ 1, \text{ if } i \neq j \text{ and points } i \text{ and } j \text{ are collinear} \\ 0, \text{ else} \end{cases}.$$

Now let $C = B-(q+1)I$ so that $C_{i,j} = 1$ if points i,j are distinct and collinear and $C_{i,j} = 0$ otherwise. It follows from the properties of generalized quadrangles in [vM] that

$$(C^2)_{i,j} = \begin{cases} q(q+1), \text{ if } i=j \\ q-1, \text{ if } i \neq j \text{ and points } i \text{ and } j \text{ are collinear} \\ q+1, \text{ else} \end{cases}.$$

It follows that $C^2 = (q+1)J + (q^2-1)I - 2C$. Thus $B^2 = 2qB + (q+1)J$. Since $GQ(q)$ is $(q+1)$-regular, $BJ = JB = (q+1)^2 J$. Thus $B^3 = 2qB^2 + (q+1)^3 J$ and $(q+1)^2 B^2 = 2q(q+1)^2 B + (q+1)^3 J$. It follows that $B(B-2qI)(B-(q+1)^2 I) = 0$. Thus B has at most three eigenvalues: $0, 2q$, and $(q+1)^2$. We obtain the multiplicities, a, b, c, of these eigenvalues from the traces of B and B^2 as follows:

$$\begin{aligned} a+b+c &= q^3+q^3+q^2+q+1 \\ 2qb+(q+1)^2c &= \mathrm{tr}B = (q^3+q^3+q^2+q+1)(q+1) \\ (2q)^2b+(q+1)^4c &= \mathrm{tr}B^2 = (q^3+q^3+q^2+q+1)(q+1)(2q+1). \end{aligned}$$

Solving for a, b, c, we obtain:

$$a = q(q^2+1)/2, \quad b = q(q+1)^2/2, \quad c = 1. \tag{12}$$

Since $A(GQ(q))^2 = N(GQ(q))^T N(GQ(q)) \oplus N(GQ(q))N(GQ(q))^T$, we have proved the following lemma:

Lemma 4.7 *Let $GQ(q)$ be the incidence graph of a generalized quadrangle of order q and let $v = (q^4-1)/(q-1)$. Then $GQ(q)$ is a trace-minimal graph in $\mathcal{G}(2v, q+1)$ and*

$$\mathrm{ch}(GQ(q), x) = x^{q(q^2+1)}(x^2-2q)^{q(q+1)^2/2}(x^2-(q+1)^2).$$

4.4.3 Generalized hexagons

Let Γ be a generalized hexagon of order q, where q is a power of a prime. There are $v = (q^6-1)/(q-1)$ points and lines in Γ and $2v$ vertices in the incidence graph $GH(q)$ of Γ. Let $B = N(GH(q))^T N(GH(q))$, where $N(GH(q))$ is the line-point incidence matrix for Γ. Using arguments similar to those in [vM, Appendix A], we get

$$B_{i,j} = \begin{cases} q+1, \text{ if } i=j \\ 1, \text{ if } i \neq j \text{ and points } i \text{ and } j \text{ are collinear} \\ 0, \text{ else.} \end{cases}$$

Now let $C = B - (q+1)I$ so that $C_{i,j} = 1$ if points i, j are distinct and collinear and $C_{i,j} = 0$ otherwise. It follows from the properties of the generalized hexagons in [vM] that

$$\left(C^2\right)_{i,j} = \begin{cases} q(q+1), \text{ if } i = j \\ q-1, \text{ if points } i \text{ and } j \text{ are collinear} \\ 1, \text{ if } i, j \text{ are not collinear and not opposite} \\ 0, \text{ if } i, j \text{ are not collinear and opposite.} \end{cases}$$

and

$$\left(C^3\right)_{i,j} = \begin{cases} (q-1)q(q+1), \text{ if } i = j \\ 2(q+1)q + (q-1)(q-2) - 1, \text{ if points } i \text{ and } j \text{ are collinear} \\ 2(q-1), \text{ if } i, j \text{ are not collinear and not opposite} \\ q+1, \text{ if } i, j \text{ are not collinear and opposite.} \end{cases}$$

It follows that

$$C^3 - (q-3)C^2 - (q+1)J - (2q-1)(q+1)I = (2q^2 + 2q - 3)C,$$

and so

$$B^3 - 4qB^2 + 3q^2 B = (q+1)J.$$

Since $JB = BJ = (q+1)^2 J$, we get

$$B^4 - 4qB^3 + 3q^2 B^2 = (q+1)^3 J.$$

Thus

$$B(B - 3qI)(B - qI)(B - (q+1)^2 I) = 0.$$

Therefore B has at most four eigenvalues: $0, q, 3q$, and $(q+1)^2$. The multiplicities, a, b, c, d of these eigenvalues are computed from traces as follows:

$$\begin{array}{rcccl} a + b + c + d & = & v & = & \frac{q^6-1}{q-1} \\ qb + 3qc + (q+1)^2 d & = & \mathrm{tr} B & = & \frac{q^6-1}{q-1}(q+1) \\ q^2 b + +9q^2 c + (q+1)^4 d & = & \mathrm{tr} B^2 & = & \frac{q^6-1}{q-1}(q+1)(2q+1) \\ q^3 b + 27q^3 c + (q+1)^6 d & = & \mathrm{tr} B^3 & = & \frac{q^6-1}{q-1}(q+1)(5q^2+4q+1). \end{array}$$

Solving for a, b, c, d we obtain

$$\begin{array}{rcl} a & = & q(q^2+q+1)(q^2-q+1)/3 \\ b & = & q(q^2-q+1)(q+1)^2/2 \\ c & = & q(q+1)^2(q^2+q+1)/6 \\ d & = & 1. \end{array} \tag{13}$$

Since the characteristic polynomial of $A(GH(q))$ is $\det(x^2 I - B)$, we have the following lemma:

Lemma 4.8 *Let $GH(q)$ be the incidence graph of a generalized hexagon of order q and let $v = (q^6 - 1)/(q-1)$. Then $GH(q)$ is a trace-minimal graph in $\mathcal{G}(2v, q+1)$ and*

$$\mathrm{ch}(GH(q), x) = x^{2a}(x^2 - q)^b (x^2 - 3q)^c (x^2 - (q+1)^2),$$

where a, b, c are given in Equation (13).

4.5 Cages

Let g, δ be positive integers. A *cage* is a δ-regular graph with girth g and a minimal number $v(g, \delta)$ of vertices. It is clear from the definition of trace-minimality that if there is a cage in a graph class $\mathcal{G}(v, \delta)$, that is there is a girth g such that $v = v(g, \delta)$, then every trace-minimal graph in $\mathcal{G}(v, \delta)$ must be a cage. Thus we have the following lemma:

Lemma 4.9 *If a graph class $\mathcal{G}(v, \delta)$ contains a cage, then every trace-minimal graph in $\mathcal{G}(v, \delta)$ is a cage. In particular, if G is the unique cage in $\mathcal{G}(v, \delta)$, then G is trace-minimal.*

There are only five known infinite families of cages: For any v, K_v (girth 3) and $K_{v,v}$ (girth 4), for any q power of prime $PP(q)$ (girth 6), $GQ(q)$ (girth 8), and $GH(q)$ (girth 12). We have seen in previous sections that all of these graphs are trace-minimal. Apart from these infinite families, cages are known for only ten pairs of values (g, δ). In the next lemmas we list the trace-minimal graphs obtained from these pairs. Precise descriptions off all these graphs and information about their discoveries can be found in [Gor].

Lemma 4.10 *The following cages (g, δ) are unique in $G(v(g, \delta), \delta)$. Thus they are trace-minimal.*

Name	g	δ	$v(g,\delta)$
Petersen	5	3	10
Robertson	5	4	19
O'Keefe-Wong	5	6	40
Hoffman-Singleton	5	7	50
O'Keefe-Wong	6	7	90
McGee	7	3	24
Balaban/McKay-Saager	11	3	112

The characteristic polynomials of the previous graphs are:

Name	Characteristic Polynomials
Petersen	$(x-3)(x-1)^5(x+2)^4$
Robertson	$(x-4)(x-1)^2(x^2-3)^2(x^2+x-5)(x^2+x-4)^2(x^2+x-3)^2(x^2+x-1)$
O'Keefe-Wong (5,6)	$(x-6)(x-2)^{18}(x-1)^4(x+2)^5(x+3)^{12}$
Hoffman-Singleton	$(x-7)(x-2)^{28}(x+3)^{21}$
O'Keefe-Wong (6,7)	$(x-7)(x-2)^{14}(x+2)^{14}(x+7)(x^2-7)^{30}$
McGee	$(x-3)(x-2)^3x^3(x+1)^2(x+2)(x^2+x-4)(x^3+x^2-4x-2)^4$
Balaban/McKay-Myrvold	$(x-3)x^{12}(x^2-6)^{15}(x^2-2)^{12}(x^3-x^2-4x+2)^2(x^3+x^2-6x-2)\times$
	$(x^4-x^3-6x^2+4x+4)^4(x^5+x^4-8x^3-6x^2+12x+4)^8$

Lemma 4.11 *The following cages (g, δ) are trace-minimal (see Figure 1).*

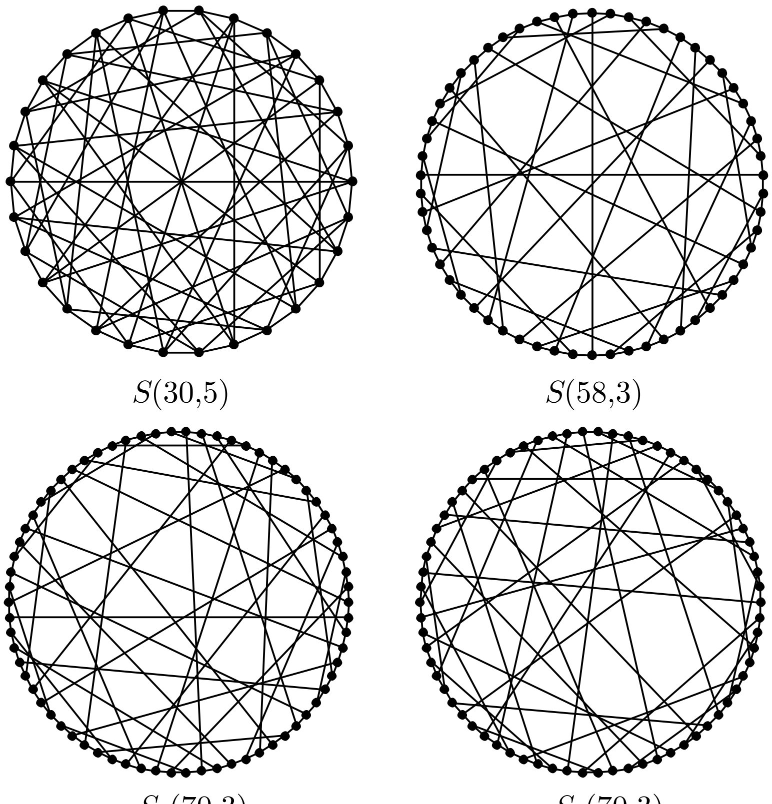

Figure 1: Trace-minimal cages

Graph	g	δ	$v(g,\delta)$	*Number of cages*
$S(30,5)$	*5*	*5*	*30*	*4*
$S(58,3)$	*9*	*3*	*58*	*18*
$S_1(70,3), S_2(70,3)$	*10*	*3*	*70*	*3*

Proof: Robertson,Wegner, Wong, Foster, and Yang, Zhang showed that there are only four cages with parameters $(5,5)$. There are only 18 cages with parameters $(9,3)$ (Brinkmann, McKay, Saager) and three cages with parameters $(10,3)$ (O'Keefe and Wong). Using the descriptions in [Gor], we calculated the spectra of each of them and found the graph that is trace-minimal in each graph class. In the case of $(10,3)$ there are two nonisomorphic trace-minimal cages in $\mathcal{G}(70,3)$. ∎

The characteristic polynomials of the previous graphs are:

Graph	Characteristic Polynomials
$S(30,5)$	$(x-5)(x-2)^8(x+1)(x+3)^4(x^4+2x^3-6x^2-7x+11)\times$ $(x^4+2x^3-4x^2-5x+5)^2$
$S(58,3)$	$(x-3)(x-1)(x+2)(x^4-x^3-4x^2+x+2)(x^5+3x^4-5x^3-17x^2+9)\times$ $(x^{10}-16x^8+x^7+88x^6-6x^5-192x^4+6x^3+141x^2-8x-24)\times$ $(x^{18}-25x^{16}+x^{15}+254x^{14}-17x^{13}-1351x^{12}+116x^{11}+4054x^{10}-$ $427x^9-6942x^8+932x^7+6607x^6-1122x^5-3209x^4+654x^3+626x^2-136x-13)^2$
$S_1(70,3)$, $S_2(70,3)$	$(x-3)(x-1)^4(x+1)^4(x+3)(x^2-6)(x^2-2)(x^4-6x^2+2)^5\times$ $(x^4-6x^2+3)^4(x^4-6x^2+6)^5$

4.6 Sporadic trace-minimal graphs

Some trace-minimal graphs that do not fit into any of the previous categories are listed here along with the graph class, characteristic polynomial. The notation for the *sporadic* trace-minimal graph in the graph class $\mathcal{G}(v,\delta)$ is $S(v,\delta)$ and for the bipartite-trace-minimal graph in $\mathcal{B}(2v,\delta)$ is $SB(2v,\delta)$.

We begin with cubic graphs. Figure 2 shows the trace-minimal cubic graphs in graph class $\mathcal{G}(n,3)$ for all even values of n from 8 to 30. Four of the cubic graphs, $n=10,14,24,30$ are also in other families of graphs known to be trace-minimal. For example the Petersen graph in $\mathcal{G}(10,3)$ is strongly regular. Three other cubic graphs, $S(58,3)$, $S_1(70,3)$, and $S_2(70,3)$ are cages and are shown in Figure 1. With the exception of $\mathcal{G}(70,3)$, each of the sporadic cubic graphs shown is the unique trace-minimal graph in its graph class.

The maximum girth and the number of nonisomorphic cubic graphs with maximum girth is known [RW] for cubic graph classes with 8, 12, 16, 18, 20, 22, 26, and 28 vertices. An algorithm that generates each of the nonisomorphic cubic graphs appears in [Bri]. Thus to determine which graph with maximum girth is trace-minimal in each graph class, we simply compute the trace sequence for each one and select the one least in lexicographic order. (We give an independent proof for the class $\mathcal{G}(8,3)$ in Lemma 6.1.) The characteristic polynomials are given in the table below:

graph class	maximum girth	number of graphs	charactersitic polynomial
$\mathcal{G}(8,3)$	4	2	$(x-3)(x-1)^2(x+1)(x^2+2x-1)^2$
$\mathcal{G}(12,3)$	5	2	$(x-3)(x-1)^2x(x+2)^2(x^2-2)^2(x^2+x-4)$
$\mathcal{G}(16,3)$	6	1	$(x-3)(x-1)^3(x+1)^3(x+3)(x^2-3)^4$
$\mathcal{G}(18,3)$	6	5	$(x-3)(x-1)(x+2)^2(x^2-x-1)^4(x^3+2x^2-4x-6)^2$
$\mathcal{G}(20,3)$	6	32	$(x-3)(x^2-x-1)^4(x^2+3x+1)(x^3-4x+1)(x^3+2x^2-4x-7)^2$
$\mathcal{G}(22,3)$	6	385	$(x-3)(x-2)^2(x-1)x(x+1)^3(x^2+x-4)(x^3+x^2-4x-2)^4$
$\mathcal{G}(26,3)$	7	3	$(x-3)(x-2)^5x^4(x+1)^2(x+2)^3(x^2+2x-2)(x^3+x^2-4x-2)^3$
$\mathcal{G}(28,3)$	7	21	$(x-3)(x-2)^5x^6(x+2)^5(x^2-2)(x^3+x^2-6x-2)(x^3+x^2-4x-2)^2$

Other sporadic graphs are shown in Figure 3. We prove these are trace-minimal in Section 6.

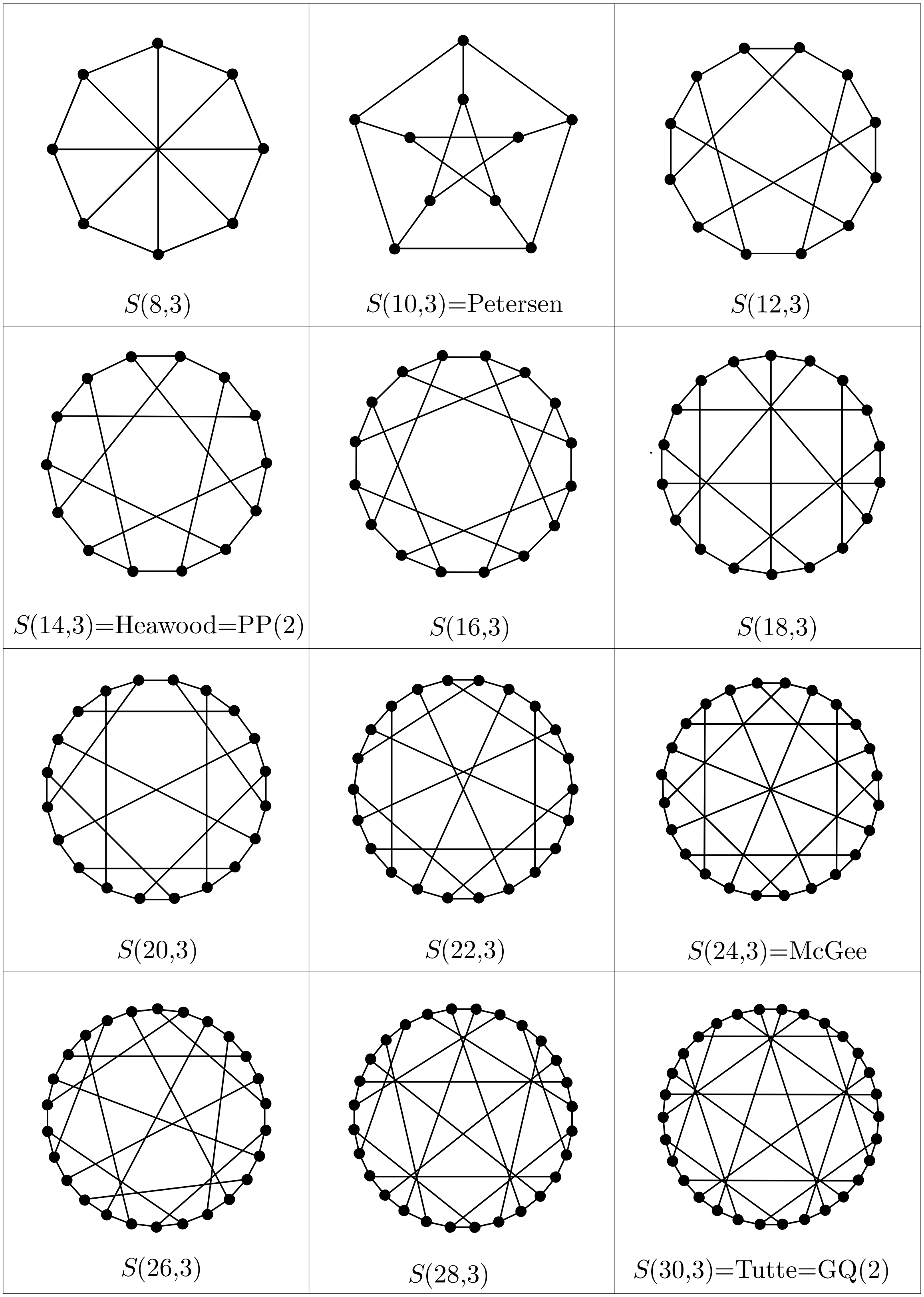

Figure 2: Sporadic Cubic Graphs

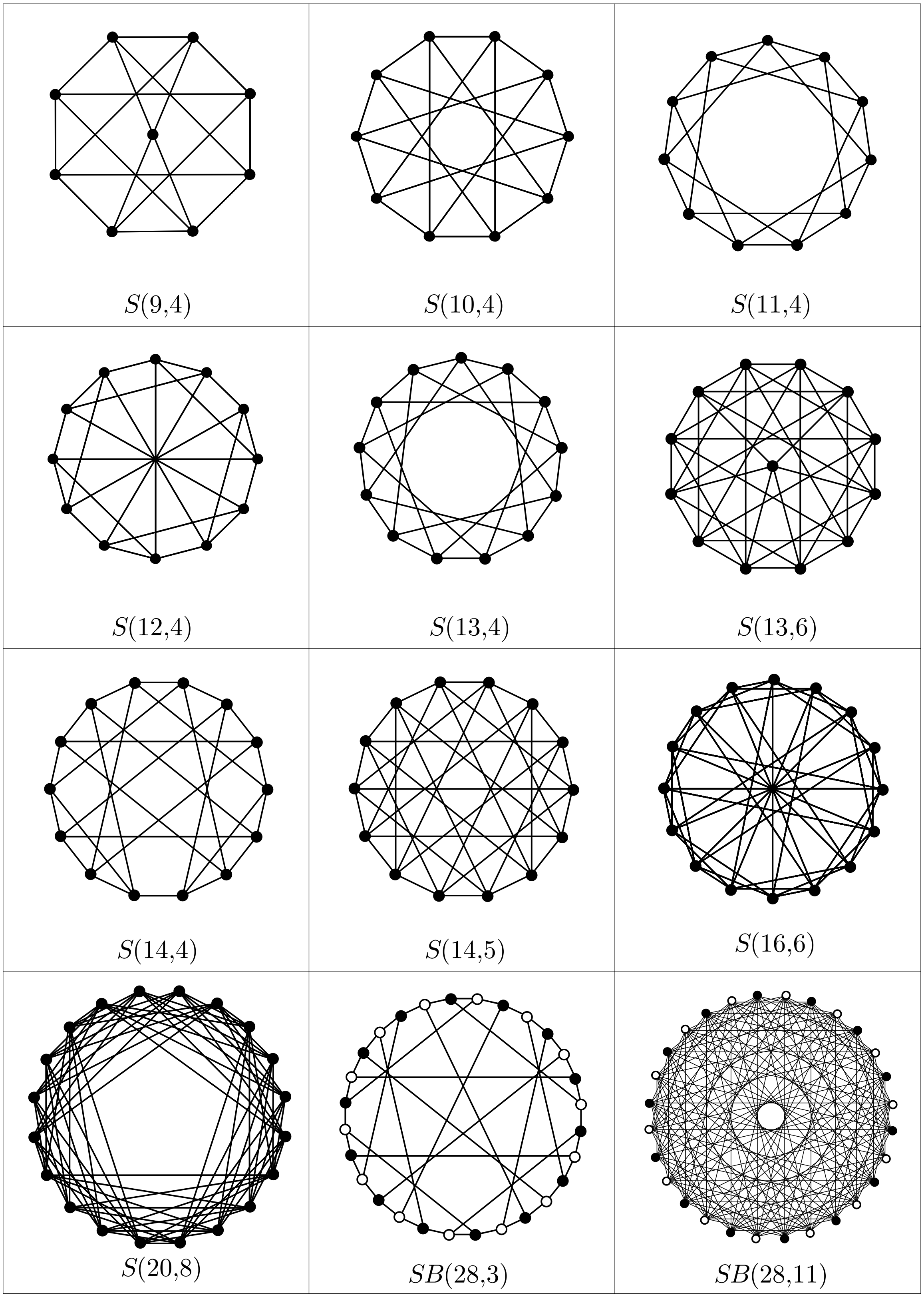

Figure 3: Sporadic Graphs, $SB(28, 11)$ is the bipartite complement of $SB(28, 3)$.

The characteristic polynomials for the sporadic graphs in Figure 3 are as follows:

graph G	$\mathrm{ch}(G,x)$
$S(9,4)$	$(x-4)(x-1)^2x^2(x+1)(x+2)(x^2+3x-2)$
$S(10,4)$	$(x-4)x^5(x^2+2x-4)^2$
$S(11,4)$	$(x-4)(x^{+}2x^4-5x^3-2x^2+4x-1)^2$
$S(12,4)$	$(x-4)(x-1)^6x(x+2)^2(x+3)^2$
$S(13,4)$	$(x^3+x^2-4x+1)^4(x-4)$
$S(13,6)$	$(x-6)(x-1)^2(x^4)(x+1)^2(x^2+x-3)(x^2+5x-3)$
$S(14,4)$	$(x-4)(x-1)^3x^2(x+1)(x+3)(x^2+x-4)^3$
$S(14,5)$	$(x-5)(x-2)(x-1)^2x^4(x+1)(x+2)(x^2+3x-6)(x^2+3x-2)$
$S(16,6)$	$(x-6)(x-2)^2x^8(x+2)(x^2+4x-4)^2$
$S(20,8)$	$(x-8)x^{15}(x^2+4x-16)^2$
$SB(28,3)$	$(x-3)(x-2)^5x^6(x+2)^5(x^2-2)(x^3+x^2-6x-2)(x^3+x^2-4x-2)^2$
$SB(28,11)$	$(x-11)(x-1)(x+1)(x+11)(x^3-4x-1)^4(x^3-4x+1)^4$

4.7 Middle values of δ

In this section we exhibit trace-minimal graphs in $\mathcal{G}(2v,\delta)$ for $\delta = v, v-1, v-2$.

4.7.1 $\mathcal{G}(2v,v)$

Theorem 4.12 *Let $v \geq 1$ be an integer. Then $K_{v,v}$ is the only trace-minimal graph in $\mathcal{G}(2v,v)$ and the only bipartite-trace-minimal graph in $\mathcal{B}(2v,v)$. And $\mathrm{ch}(K_{v,v},x) = (x-v)(x+v)x^{2v-2}$*

Proof: The complete bipartite graph $K_{v,v}$ is the only graph in $\mathcal{B}(2v,v)$ so it must be the only bipartite-trace-minimal graph in $\mathcal{B}(2v,v)$. But $K_{v,v}$ is also the only trace-minimal graph in $\mathcal{G}(2v,v)$. To see this let G be a trace-minimal graph in $\mathcal{G}(2v,v)$. Since $K_{v,v}$ has no 3-cycles, then by Theorem 2.1 G has no 3-cycles. Assume that vertex 1 is adjacent to vertices $v+1,\ldots,2v$. Since G has no 3-cycles, none of the vertices $v+1,\ldots,2v$ are adjacent to each other. Thus each of the vertices $v+1,\ldots,2v$ is adjacent to each of the vertices $1,2,\ldots,v$. That is, $G = K_{v,v}$. ∎

4.7.2 $\mathcal{G}(2v,v-1)$ and $\mathcal{G}(2v,v-2)$

To deal with the classes $\mathcal{G}(2v,v-1)$ and $\mathcal{G}(2v,v-2)$, we need the following lemma:

Lemma 4.13 *Let v be a positive integer.*

1. Let $v \geq 6$ and G be a graph in $\mathcal{G}(2v,v-1)$ with no 3-cycles. Then G is bipartite.

2a. Let $v \geq 7$ and G be a graph in $\mathcal{G}(2v,v-2)$ with no 3-cycles and no 5-cycles. Then G is bipartite.

2b. Let $v \geq 11$ and G be a graph in $\mathcal{G}(2v,v-2)$ with no 3-cycles. Then G is bipartite.

Proof: Assume G is not bipartite and has no 3-cycles. Then G has an odd cycle of length at least 5. Let $1, \ldots, m$ be the vertices of the shortest odd cycle in G. No two vertices i, j among $1, \ldots, m$ are adjacent unless $i - j = \pm 1 \pmod{m}$ since any other edge would become part of a shorter odd cycle. For each $i = 1, \ldots, m$, let A_i be the set of vertices adjacent to vertex i excluding vertices $i \pm 1$ in the cycle. Clearly, $A_i \cap A_j = \emptyset$ unless $i = j$ or $i - j \equiv \pm 2 \pmod{m}$, since any such nonempty intersection would be part of a shorter odd cycle.

1. Let $G \in \mathcal{G}(2v, v-1)$. In this case $|A_i| = v-3$ for all i. Since $A_1 \cap A_2 = \emptyset$, we have $2v \geq m + |A_1 \cup A_2| = m + 2(v-3)$ and hence $m \leq 6$. It follows that $m = 5$. Next we show that $|A_1 \cap A_3| \geq v - 4$. Since $A_1 \cap A_2 = A_2 \cap A_3 = \emptyset$,

$$2v - 5 \geq |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_3| = 3(v-3) - |A_1 \cap A_3|.$$

Similarly $|A_3 \cap A_5| \geq v - 4$. Since A_1 and A_5 are disjoint,

$$v - 3 = |A_3| \geq |A_1 \cap A_3| + |A_3 \cap A_5| = 2(v-4).$$

Thus $v \leq 5$.

2. Let $G \in \mathcal{G}(2v, v-2)$. In this case $|A_i| = v - 4$ for all i. Arguing as above, we get $2v \geq m + 2(v-4)$ so that $m \leq 8$. Thus $m = 5$ or $m = 7$.

Consider first the case $m = 7$. Since $A_1 \cap A_2 = A_2 \cap A_3 = \emptyset$,

$$2v - 7 \geq |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_3| = 3(v-4) - |A_1 \cap A_3|.$$

Thus $|A_1 \cap A_3| \geq v - 5$ and likewise $|A_3 \cap A_5| \geq v - 5$. We have $A_1 \cap A_5 = \emptyset$. So

$$v - 4 = |A_3| \geq |A_1 \cap A_3| + |A_3 \cap A_5| \geq 2(v-5).$$

It follows that $v \leq 6$.

Now consider the case $m = 5$. In this case, $|A_1 \cap A_3| \geq v - 7$. The reason is that $A_1 \cap A_2 = A_2 \cap A_3 = \emptyset$. Thus

$$2v - 5 \geq |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_3| = 3(v-4) - |A_1 \cap A_3|.$$

Likewise, $|A_3 \cap A_5| \geq v - 7$.

Now since $A_1 \cap A_5 = \emptyset$, we have

$$v - 4 = |A_3| \geq |A_1 \cap A_3| + |A_3 \cap A_5| \geq 2(v-7).$$

So $v \leq 10$. ∎

Theorem 4.14 *Let $v \geq 1$ be an integer. The trace-minimal graph in $\mathcal{G}(2v, v-1)$ is unique and is given (along with $ch(G,x)$) as follows:*

v	*graph* G	$ch(G,x)$
$v \neq 4,5$	$K_{v,v} - vK_2$	$(x^2 - (v-1)^2)(x^2-1)^{v-1}$
$v = 4$	$S(8,3)$	$(x-3)(x-1)^2(x+1)(x^2+2x-1)^2$
$v = 5$	$S(10,4)$	$(x-4)x^5(x^2+2x-4)^2$

Proof: Let G be a trace-minimal graph in $\mathcal{G}(2v, v-1)$. The graph $K_{v,v} - vK_2 \in \mathcal{G}(2v, v-1)$ is bipartite and has no 3-cycles. Thus G has no 3-cycles by Theorem 2.1. If $v \geq 6$, it follows from Lemma 4.13 that G is bipartite. But the only bipartite graph in $\mathcal{G}(2v, v-1)$ is $K_{v,v} - vK_2$ so $G = K_{v,v} - vK_2$.

If $v = 1, 2$ the only graph in $\mathcal{G}(2v, v-1)$ is $K_{v,v} - vK_2$ so again $G = K_{v,v} - vK_2$. The only remaining cases are $v = 3, 4, 5$. But $C_6 = K_{3,3} - 3K_2$ was shown to be trace-minimal in $\mathcal{G}(6, 2)$ in Lemma 4.2. The cases $v = 4, 5$ are proved in Lemmas 6.1 and 6.3.

It is easy to see that the adjacency matrix for $K_{v,v} - vK_2$ is a 2×2 block matrix in which the diagonal blocks are zero and the off-diagonal blocks are $J - I$. The computation for the characteristic polynomial is routine. ∎

Theorem 4.15 *Let $v \geq 2$ be an integer. The trace-minimal graph G in $\mathcal{G}(2v, v-2)$ is unique.*

If $v \neq 5, 6, 7, 8, 10$, then $G = K_{v,v} - C_{2v}$ with characteristic polynomial

$$\mathrm{ch}(G, x) = \frac{x^2 - (v-2)^2}{x^2 - 4} \left(2\mathrm{Tch}_{2v}(x/2) - 2\right).$$

For $v = 5, 6, 7, 8, 10$, the trace-minimal graph G and its characteristic polynomial are given in the table:

v	*graph G*	$\mathrm{ch}(G, x)$
$v = 5$	*Petersen*	$(x-3)(x-1)^5(x+2)^4$
$v = 6$	$S(12, 4)$	$(x-4)(x-1)^6 x(x+2)^2(x+3)^2$
$v = 7$	$S(14, 5)$	$(x-5)(x-2)(x-1)^2 x^4 (x+1)(x+2)(x^2+3x-6)(x^2+3x-2)$
$v = 8$	$S(16, 6)$	$(x-6)(x-2)^2 x^8 (x+2)(x^2+4x-4)^2$
$v = 10$	$S(20, 8)$	$(x-8)x^{15}(x^2+4x-16)^2$

Proof: Let G be a trace-minimal graph in $\mathcal{G}(2v, v-2)$. The graph $K_{v,v} - C_{2v} \in \mathcal{G}(2v, v-2)$ is bipartite and has no 3-cycles. Thus G has no 3-cycles. If $v \geq 11$, it follows from Lemma 4.13 that G is bipartite. But from Lemma 4.2 the only bipartite-trace-minimal graph in $\mathcal{G}(2v, v-2)$ is $K_{v,v} - C_{2v}$. Thus $G = K_{v,v} - C_{2v}$.

For $v \leq 5$, $I_4 = K_{2,2} - C_4$, $3K_2 = K_{3,3} - C_6$, $C_8 = K_{4,4} - C_8$, and the Petersen graph, were proved to be trace-minimal in Lemmas 4.1, 4.2, and 4.3.

The cases $v = 6, 7, 8, 10$ are analyzed in Lemma 6.4. Finally, the proof that $K_{9,9} - C_{18}$ is trace-minimal in $\mathcal{G}(18, 7)$ is given in Lemma 6.5. ∎

4.8 Large values of δ

In this section we exhibit trace-minimal graphs in $\mathcal{G}(v, \delta)$ for all v and $v - 6 \leq \delta \leq v - 1$. (Note that for $\delta = v - 2$, $v - 4$, $v - 6$, v must be even.)

4.8.1 $\delta = v-1, v-2$

See Lemma 4.1.

4.8.2 $\delta = v-3$

Lemma 4.16 *Let $v \geq 3$ be an integer. The trace-minimal graph G in $\mathcal{G}(v, v-3)$ is unique and is given (along with* $\mathrm{ch}(G,x)$*) as follows:*

$v \pmod 3$	*graph G*	$\mathrm{ch}(G,x)$
$v = 3l$	$I_3^{(l)}$	$(x-(v-3))x^{2l}(x+3)^{l-1}$
$v = 3l+1$	$2K_2 \bigtriangledown I_3^{(l-1)}$	$(x-(v-3))x^{2l-2}(x+3)^{l-1}(x+1)^2(x-1)$
$v = 3l+2$	$C_5 \bigtriangledown I_3^{(l-1)}$	$(x-(v-3))x^{2l-2}(x+3)^{l-1}(x^2+x-1)^2.$

4.8.3 $\delta = v-4$

Lemma 4.17 *Let $v \geq 4$ be an even integer. The trace-minimal graph G in $\mathcal{G}(v, v-4)$ is unique and is given (along with* $\mathrm{ch}(G,x)$*) as follows:*

$v \pmod 4$	*graph G*	$\mathrm{ch}(G,x)$
$v = 4l$	$I_4^{(l)}$	$(x-(v-4))x^{3l}(x+4)^{l-1}$
$v = 4l+2$	$C_6 \bigtriangledown I_4^{(l-1)}$	$(x-(v-4))(x-1)^2x^{3l-3}(x+1)^2(x+2)(x+4)^{l-1}.$

4.8.4 $\delta = v-5$

Lemma 4.18 *Let $v \geq 5$ be an integer. The trace-minimal graph G in $\mathcal{G}(v, v-5)$ is unique and is given (along with $ch_G(x)$) as follows:*

v	*graph G*	$ch(G,x)$
$v = 5l$	$I_5^{(l)}$	$(x-(v-5))x^{4l}(x+5)^{l-1}$
$v = 5l+1$	$3K_2 \bigtriangledown I_5^{(l-1)}$	$(x-(v-5))(x-1)^2x^{4l-4}(x+1)^3(x+5)^{l-1}$
$v = 5l+2$	$C_7 \bigtriangledown I_5^{(l-1)}$	$(x-(v-5))x^{4l-4}(x+5)^{l-1}(x^3+x^2-2x-1)^2$
$v = 5l+3$	$S(8,3) \bigtriangledown I_5^{(l-1)}$	$(x-(v-5))(x-1)^2x^{4l-4}(x+1)(x+5)^{l-1}(x^2+2x-1)^2$
$v = 5l+4$	$S(9,4) \bigtriangledown I_5^{(l-1)}$	$(x-(v-5))(x-1)^2x^{4l-2}(x+1)(x+2)(x+5)^{l-1}(x^2+3x-2)$

4.8.5 $\delta = v-6$

Lemma 4.19 *Let $v \geq 6$ be an even integer. The trace-minimal graph G in $\mathcal{G}(v, v-6)$ is unique and is given (along with $ch(G,x)$) as follows:*

v	*graph G*	$ch_G(x)$
$v = 6l$	$I_6^{(l)}$	$(x-(v-6))x^{5l}(x+6)^{l-1}$
$v = 6l+2$	$C_8 \bigtriangledown I_6^{(l-1)}$	$(x-(v-6)x^{5l-3}(x+2)(x+6)^{l-1}(x^2-2)^2$
$v = 6l+4$	$S(10,4) \bigtriangledown I_6^{(l-1)}$	$(x-(v-6))x^{5l}(x+6)^{l-1}(x^2+2x-4)^2$

4.9 Proofs for large δ

Many of the trace-minimal graphs in Section 4.8 are the unique graph in $\mathcal{G}(v,\delta)$ with the fewest number of triangles and hence trace-minimal by Theorem 2.1. Letting $\triangle(G)$ denote the number of triangles (3-cycles) in G, it is easy to see that

$$\triangle(G) = \frac{1}{6}\mathrm{tr}A(G)^3.$$

It is useful to notice that if $G \in \mathcal{G}(v,\delta)$ and $G^{\mathrm{comp}} \in \mathcal{G}(v,\delta')$ is the complement of G ($\delta + \delta' = v-1$), then

$$\mathrm{tr}A(G)^3 + \mathrm{tr}A(G^{\mathrm{comp}})^3 = 6\binom{v}{3} - 3v\delta\delta'.$$

Thus G has the fewest number of triangles in $\mathcal{G}(v,\delta)$ if and only if G^{comp} has the largest number of triangles in $\mathcal{G}(v,\delta')$. In the following sections, it is convenient to deal with graphs having the largest number of triangles instead of the smallest number of triangles. We denote the minimum and maximum number of triangles in a graph class as follows:

$$\begin{aligned}\min\triangle(v,\delta) &= \mathrm{minimum}\{\triangle(G) : G \in \mathcal{G}(v,\delta)\}\\ \max\triangle(v,\delta) &= \mathrm{maximum}\{\triangle(G) : G \in \mathcal{G}(v,\delta)\}\end{aligned}$$

By the comments above, we have

$$\min\triangle(v,\delta) + \max\triangle(v,\delta') = \binom{v}{3} - \frac{1}{2}v\delta\delta', \tag{14}$$

where $\delta + \delta' = v - 1$.

In the next few results, we shall see that for $\delta' \leq 5$, all but one component of the graph in $\mathcal{G}(v,\delta')$ with the largest number of triangles are the complete graph $K_{\delta'+1}$. And that the other component H has fewer than $2\delta'+2$ vertices. Thus, by the remarks above, the graph in $\mathcal{G}(v,\delta)$ with the fewest number of triangles is of the form $(H + kK_{\delta'+1})^{\mathrm{comp}} = H^{\mathrm{comp}} \bigtriangledown I_{\delta'+1}^{(k)}$, for some nonnegative integer k.

Let i be a vertex of a graph G and define $\triangle(G,i)$ to be the number of triangles in G in which i is a vertex. Likewise if (i,j) is an edge in G, let $\triangle(G,(i,j))$ denote the number of triangles in G that include edge (i,j).

Theorem 4.20 *Let $\delta \geq 3$ and $G \in \mathcal{G}(v,\delta)$. If $\triangle(G) = \max\triangle(v,\delta)$ then either $K_{\delta+1}$ is a component of G or $\triangle(G,x) \leq \binom{\delta}{2} - 2$ for all vertices x in G.*

Proof: Assume $G \in \mathcal{G}(v,\delta)$, $\triangle(G) = \max\triangle(v,\delta)$, and $\triangle(G,u) > \binom{\delta}{2} - 2$ for some vertex u in G. Let N be the subgraph of $\delta+1$ vertices consisting of u and all vertices adjacent to u. If $\triangle(G,u) = \binom{\delta}{2}$, then $N = K_{\delta+1}$ is a component of G. Thus we shall assume that $\triangle(G,u) = \binom{\delta}{2} - 1$. Then every pair of vertices in N are adjacent except one pair, say vertex 1 and vertex 2. Thus no vertex in N is adjacent to a vertex not in N, except for vertex 1 and vertex 2. However, each of these two vertices must be adjacent to exactly one vertex that is not in N. If vertices 1 and 2 are adjacent to the same vertex not in N, we label it vertex 3. If they are adjacent to different vertices not in N, we shall label these vertices 3 and 4 and assume that (1,3) and (2,4) are edges in G.

In either case, we construct a graph $H \in \mathcal{G}(v,\delta)$ by removing at least the two edges, (1,3) and (2,3), or (1,3) and (2,4), from G and adding edge (1,2) and others to obtain H. Since $\triangle(G,(1,3)) = \triangle(G,(2,4)) = 0$, no triangles are lost from G by removing these two edges. However, when we add edge (1,2) to get H, $K_{\delta+1}$ is a component of H and we gain $\triangle(H,(1,2)) = \delta - 1$ triangles. In each of the four cases below, the number of triangles in H exceeds the number of triangles in G, contradicting the assumption that $\triangle(G) = \max\triangle(v,\delta)$.

Case 1: (1,3) and (2,4) are edges, and (3,4) is not an edge. (See Figure 4.) Let $H \in \mathcal{G}(v,\delta)$ be the graph obtained from G by deleting edges (1,3) and (2,4) and adding edges (1,2) and (3,4). Then H has at least $\delta - 1$ more triangles than G.

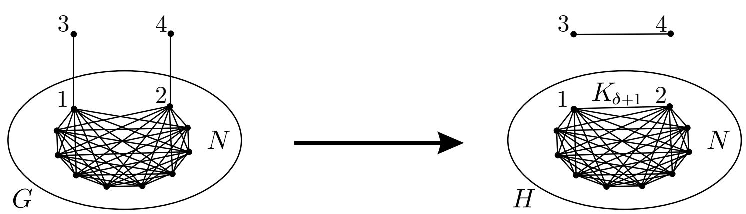

Figure 4: (3,4) is not an edge of G

Case 2: (1,3), (2,4), and (3,4) are edges, and there is a vertex, say 5, adjacent to exactly one of the vertices 3,4. Say (3,5) is an edge but (4,5) is not an edge. (See Figure 5.) Delete edges (1,3), (2,4), (5,6) from G and add edges (1,2), (3,6), (4,5) to obtain a graph $H \in \mathcal{G}(v,\delta)$. Since vertices 5 and 6 can have at most $\delta - 1$ common neighbors, $\triangle(G,(5,6)) \le \delta - 1$. But vertices 3,4,5 form a triangle in H and hence, H has at least one more triangle than G.

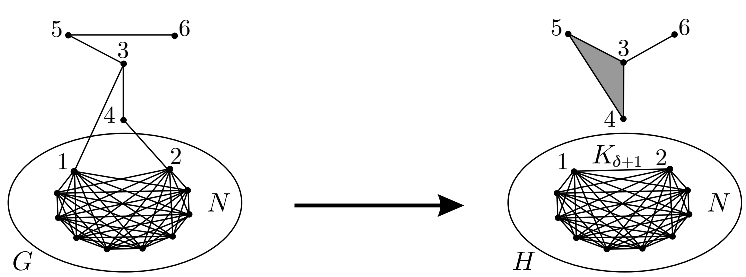

Figure 5: vertex 5 in G is adjacent to 3 but not to 4

Case 3: (1,3), (2,4), and (3,4) are edges, but there is no vertex adjacent to exactly one of 3,4. In this case all $\delta - 2$ neighbors of vertex 3, except vertex 1 and vertex 4 are common neighbors of vertex 4. (See Figure 6.) Let vertex 5 be one of these. Then vertex 5 must be adjacent to a vertex, 6, that is not adjacent to either vertex 3 or 4. And there must be a vertex, 7, adjacent to vertex 6, but not to vertex 3 or vertex 4. In particular, neither (3,6) nor (4,7) is an edge in G. Now delete edges (1,3), (2,4) and (6,7) from G and add edges (1,2), (3,6), and (4,7) to obtain H. As before, the deletion of edge (6,7) results in a loss of at most $\delta - 1$ triangles whereas $\triangle(H,(1,2)) = \delta - 1$ and vertices 3,5,6 form a triangle in H but not in G. Thus H has at least one more triangle than G.

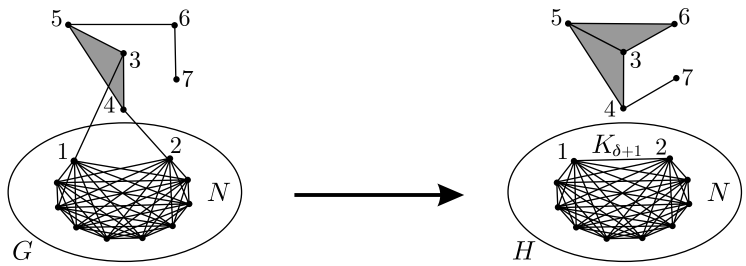

Figure 6: no vertex in G is adjacent to exactly one of 3,4

Case 4: (1,3) and (2,3) are edges. Thus vertex 3 is the only neighbor of a vertex in N that is not in N. It follows that there are vertices 4, 5, and 6, not in N, such that (3,4), (4,5), and (5,6) are edges. (See Figure 7.) Delete edges (1,3), (2,3), and (5,6) and add edges (1,2), (3,5), and (3,6) to obtain H. Again the deletion of edge (5,6) results in a loss of at most $\delta - 1$ triangles whereas the addition of (1,2) gains $\delta - 1$ triangles. Also vertices 3,4,5 form a triangle in H. Thus H has at least one more triangle than G.

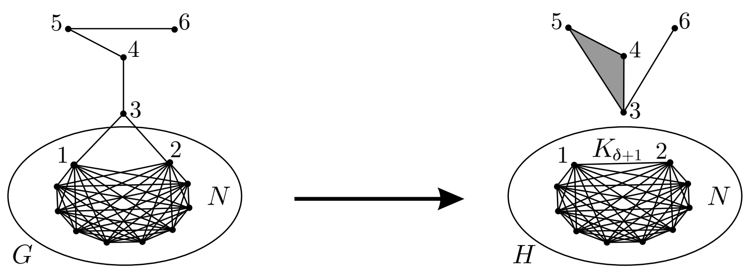

Figure 7: (1,3) and (2,3) are edges of G

∎

Corollary 4.21 *Let $\delta \geq 3$ and $G \in \mathcal{G}(v, v-\delta-1)$. Let q and ρ be nonnegative integers satisfying $v = q(\delta+1)+\rho$ and $0 \leq \rho \leq \delta$. If G is trace-minimal, then either $G = I_{\delta+1} \bigtriangledown H$ for some trace-minimal graph $H \in \mathcal{G}(v-\delta-1, v-2\delta-2)$, or*

$$v \leq \frac{\rho}{4}\left((\delta+1)^2 - \rho^2\right) + \frac{3}{2}\min\triangle(\delta+1+\rho, \rho). \tag{15}$$

Proof: Let $G \in \mathcal{G}(v, v-\delta-1)$ be trace-minimal. Then $G^{\mathrm{comp}} \in \mathcal{G}(v,\delta)$ and $\triangle(G^{\mathrm{comp}}) = \max\triangle(v,\delta)$. So by Theorem 4.20 either $K_{\delta+1}$ is a component of G^{comp} or $\triangle(G^{\mathrm{comp}}, i) \leq \binom{\delta}{2} - 2$, for all vertices i in G^{comp}. If $K_{\delta+1}$ is a component of G^{comp}, then $G = I_{\delta+1} \bigtriangledown H$ for some $H \in \mathcal{G}(v-\delta-1, v-2\delta-2)$. And by Lemma 3.2, H is trace-minimal.

Otherwise, $\triangle(G^{\mathrm{comp}}, i) \leq \binom{\delta}{2} - 2$ for all vertices, i of G. Thus

$$\max\triangle(v,\delta) = \triangle(G^{\mathrm{comp}}) = \frac{1}{3}\sum \triangle(G^{\mathrm{comp}}, i) \leq \frac{v}{3}\left[\binom{\delta}{2} - 2\right], \tag{16}$$

where the sum is taken over all vertices, i of G^{comp}. Let $G_1 = (q-1)K_{\delta+1} + H \in \mathcal{G}(v,\delta)$, where $H \in \mathcal{G}(\delta+1+\rho,\delta)$ satisfies $\triangle(H) = \max\triangle(\delta+1+\rho,\delta)$. Then

$$\max\triangle(v,\delta) \geq \triangle(G_1) = (q-1)\binom{\delta+1}{3} + \max\triangle(\delta+1+\rho,\delta). \tag{17}$$

By Equation (14),

$$\max\triangle(\delta+1+\rho,\delta) = \binom{\delta+1+\rho}{3} - \frac{1}{2}\delta\rho(\delta+1+\rho) - \min\triangle(\delta+1+\rho,\rho). \tag{18}$$

Thus, by (16), (17), and (18), we have

$$\frac{v}{3}\left[\binom{\delta}{2} - 2\right] \geq (q-1)\binom{\delta+1}{3} + \binom{\delta+1+\rho}{3} - \frac{1}{2}\delta\rho(\delta+1+\rho) - \min\triangle(\delta+1+\rho,\rho),$$

which implies (15). ∎

We also need a result by Pullman and Wormald [PW]:

Lemma 4.22 *[PW] There exists a graph $G \in \mathcal{G}(v,\delta)$ with $\triangle(G) = 0$ if and only if $2\delta < v$ for v even or $5\delta < 2v$ for v odd.*

4.9.1 Proof of Lemma 4.16

Let $v \geq 3$ be an integer and let G be a trace-minimal graph in $\mathcal{G}(v, v-3)$. Then $\triangle(G) = \min\triangle(v, v-3)$, $G^{\mathrm{comp}} \in \mathcal{G}(v,2)$, and $\triangle(G^{\mathrm{comp}}) = \max\triangle(v,2)$. Every graph in $\mathcal{G}(v,2)$ is a direct sum of its cyclic components. Thus all or all but one of the components of G^{comp} must be 3-cycles. If $v = 3l$, then $G^{\mathrm{comp}} = lC_3$ and $G = I_3^{(l)}$. If $v = 3l+1$, then $G^{\mathrm{comp}} = C_4 + (l-1)C_3$ and $G = (C_4)^{\mathrm{comp}} \bigtriangledown I_3^{(l-1)} = 2K_2 \bigtriangledown I_3^{(l-1)}$. And if $v = 3l+2$, then $G^{\mathrm{comp}} = C_5 + (l-1)C_3$ and $G = C_5 \bigtriangledown I_3^{(l-1)}$, since $(C_5)^{\mathrm{comp}} = C_5$.

To compute the characteristic polynomials $\mathrm{ch}(G,x)$ we need $\mathrm{ch}(C_3,x) = (x-2)(x+1)^2$, $\mathrm{ch}(C_4,x) = x^2(x+2)(x-2)$ and $\mathrm{ch}(C_5,x) = (x-2)(x^2+x-1)^2$.

Suppose $v = 3l+1$. Then $G^{\mathrm{comp}} = C_4 + (l-1)C_3$. Thus $\mathrm{ch}(G^{\mathrm{comp}},x) = \mathrm{ch}(C_4,x)[\mathrm{ch}(C_3,x)]^{l-1}$ and by Equation (10)

$$\begin{aligned}\mathrm{ch}(G,x) &= \pm(x-(v-3))(-x-1)^2((-x-1)+2)((-x-1)-2)[(-x-1)-2)((-x-1)+1)^2]^{l-1}/(x+3)\\ &= (x-(v-3))(x+1)^2(x-1)(x+3)^{l-1}x^{2l-2}.\end{aligned}$$

The characteristic polynomials for other cases, $v = 3l$ and $3l+2$ are computed in a similar way.

4.9.2 Proof of Lemma 4.17

Let $v \geq 4$ be an even integer and let G be a trace-minimal graph in $\mathcal{G}(v, v-4)$. Let $v = 4l + \rho$, where $\rho = 0, 2$. We apply Corollary 4.21 with $\delta = 3$. If $\rho = 0$, then the right side of Inequality (15) is 0. Thus by repeated application of Corollary 4.21, $G = I_4^{(l-1)} \bigtriangledown H$, where H is a trace-minimal graph in $\mathcal{G}(4,0)$. But then $H = I_4$ and so $G = I_4^{(l)}$.

Now suppose $\rho = 2$. Since $\min\triangle(6,2) = 0$ (by Lemma 4.22), the right side of Inequality (15) is 6. By Corollary 4.21, $G = I_4^{(l-1)} \bigtriangledown H$, where H is a trace-minimal graph in $\mathcal{G}(6,2)$. Then by Lemma 4.2 $H = C_6$.

To compute the characteristic polynomial of $C_6 \bigtriangledown I_4^{(l-1)}$, we need $\mathrm{ch}(C_6,x) = (x-2)(x-1)^2(x+1)^2(x+2)$ from Lemma 4.2. It is easy to show that $\mathrm{ch}(I_4^{(l-1)},x) = (x-4(l-2))x^{3(l-1)}(x+4)^{l-2}$. Thus from Equation (11) we have

$$\begin{aligned}\mathrm{ch}(G,x) &= \pm\frac{(x-(4l-2))(x+4)}{(x-2)(x-4(l-2))}\mathrm{ch}(C_6,x)\mathrm{ch}(I_4^{(l-1)},x) \\ &= (x-(v-4))(x+4)^{l-1}x^{3(l-1)}(x-1)^2(x+1)^2(x+2).\end{aligned}$$

If $v = 4l$ then

$$\mathrm{ch}(G,x) = (x-(v-4))x^{3l}(x+4)^{l-1}.$$

4.9.3 Proof of Lemma 4.18

Let $v \geq 5$ be an integer and let G be a trace-minimal graph in $\mathcal{G}(v, v-5)$ and let $v = 5l + \rho$, where $\rho = 0,1,2,3,4$. We apply Corollary 4.21 with $\delta = 4$. Thus there are five cases.

First suppose $\rho = 0$. Then the right side of Inequality (15) is 0. By repeated application of Corollary 4.21, $G = I_5^{(l-1)} \bigtriangledown H$, where H is a trace-minimal graph in $\mathcal{G}(5,0)$. Thus $H = I_5$ and so $G = I_5^{(l)}$.

Now suppose $\rho = 1$. Then the right side of Inequality (15) is 6, since the matching $3K_2$ in $\mathcal{G}(6,1)$ has no triangles. Thus $G = I_5^{(l-1)} \bigtriangledown H$, where H is a trace-minimal graph in $\mathcal{G}(6,1)$. Thus $H = 3K_2$ by Lemma 4.1 and $G = I_5^{(l-1)} \bigtriangledown 3K_2$.

For $\rho = 2$ the right side of Inequality (15) is 10.5. (The graph C_7 has no triangles.) So $G = I_5^{(l-1)} \bigtriangledown H$, where H is trace-minimal in $\mathcal{G}(7,2)$. Then by Lemma 4.2, $H = C_7$.

For $\rho = 3$ the right side of Inequality (15) is 12. (The graph $S(8,3)$ has no triangles; see Figure 2.) So $G = I_5^{(l-1)} \bigtriangledown H$, where H is trace-minimal in $\mathcal{G}(8,3)$. Then by Lemma 6.1, $H = S(8,3)$.

For $\rho = 4$ the right side of Inequality (15) is 12, since $S(9,4)$ has two triangles, which is the minimum, $\min\triangle(9,4)$. (See Figure 3.) So $G = I_5^{(l-1)} \bigtriangledown H$, where H is trace-minimal in $\mathcal{G}(9,4)$. Then by Lemma 6.2, $H = S(9,4)$.

The characteristic polynomial of these graphs can be computed using Equation (11). It is not difficult to see that $\mathrm{ch}(I_5^{(l-1)},x) = (x-5(l-2))x^{4l-4}(x+5)^{l-2}$. From Section 4.6 we have $\mathrm{ch}(S(8,3),x) = (x-3)(x-1)^2(x+1)(x^2+2x-1)^2$. Now to compute the characteristic polynomial of $S(8,3) \bigtriangledown I_5^{(l-1)}$ we use Equation (11) with $G_1 = S(8,3), G_2 = I_5^{(l-1)}$. Then $v_1 = 8, \delta_1 = 3, v_2 = 5(l-1), \delta_2 = 5(l-2)$

and so $v = 5l + 3, \delta = 5l - 2$. Thus,

$$\begin{aligned} \mathrm{ch}(S(8,3) \bigtriangledown I_5^{(l-1)}, x) &= \left(\frac{\mathrm{ch}(S(8,3),x)}{x-3}\right)\left(\frac{\mathrm{ch}(I_5^{(l-1)},x)}{x-5(l-2)}\right)(x-(5l-2))(x+5) \\ &= (x-1)^2(x+1)(x^2+2x-1)^2x^{4l-4}(x+5)^{l-2}(x-(5l-2))(x+5) \\ &= (x-(5l-2))(x-1)^2x^{4l-4}(x+1)(x+5)^{l-1}(x^2+2x-1)^2. \end{aligned}$$

The characteristic polynomials for the remaining cases are computed in a similar way.

4.9.4 Proof of Lemma 4.19

Let $v \geq 6$ be an even integer. Let G be a trace-minimal graph in $\mathcal{G}(v, v-6)$ and let $v = 6l + \rho$ where $\rho = 0, 2, 4$. We apply Corollary 4.21 with $\delta = 5$.

If $\rho = 0$ then the right side of Equation (15) is 0. So by Corollary 4.21, $G = I_6^{(l-1)} \bigtriangledown H$, where H is trace-minimal in $\mathcal{G}(6, 0)$. Then $H = I_6$ and $G = I_6^l$.

In the case $\rho = 2$, the right side of Equation (15) is 16. (The graph C_8 has no triangles.) So Corollary 4.21 implies that $G = I_6^{(l-2)} \bigtriangledown H$ with H trace-minimal in $\mathcal{G}(14, 8)$. Then $H = I_6 \bigtriangledown C_8$ by Lemma 6.6 and $G = I_6^{(l-1)} \bigtriangledown C_8$.

In the case $\rho = 4$, the right side of Equation (15) is 20, since $S(10, 4)$ has no triangles. (See Figure 3.) So the Corollary 4.21 implies that $G = I_6^{(l-2)} \bigtriangledown H$ with H trace-minimal in $\mathcal{G}(16, 10)$. Then $H = I_6 \bigtriangledown S(10, 4)$ by Lemma 6.6 and $G = I^{(l-1)} \bigtriangledown S(10, 4)$.

To compute the characteristic polynomials of these graphs, first note that $\mathrm{ch}(I_6^{(l)}, x) = (x-6(l-1))x^{5l}(x+6)^{l-1}$, which follows from Equation (11) and $\mathrm{ch}(I_6, x) = x^6$. Now using Equation (11) and the characteristic polynomial of $S(10, 4)$ from Section 4.6, we have

$$\frac{\mathrm{ch}(S(10,4) \bigtriangledown I_6^{(l-1)}, x)}{(x+6)(x-(6l-2))} = \frac{(x-4)x^5(x^2+2x-4)^2}{x-4} \cdot \frac{(x-6(l-2))x^{5l-5}(x+6)^{l-2}}{x-6(l-2)},$$

and thus

$$\mathrm{ch}(S(10,4) \bigtriangledown I_6^{(l-1)}, x) = (x-(6l-2))x^{5l}(x+6)^{l-1}(x^2+2x-4)^2.$$

The characteristic polynomial of $C_8 \bigtriangledown I_6^{(l-1)}$ can be computed in a similar way.

5 Formulas for $P(n, r, t)$

In this section, we exhibit formulas for $P(n, r, t)$ for various pairs (n, r) where $n \equiv -1 \pmod 4$ and $0 \leq r < n$. In each case, the formula is obtained from a corresponding (bipartite-)trace-minimal graph and its characteristic polynomial using one of the theorems from Section 1.3. Rather than writing the routine calculation for each pair (n, r), we simply list the graph class and (bipartite-)trace-minimal graph in the class along with the section in which the graph and its characteristic polynomial are found and the theorem and equation from Section 1.3 that is used. To illustrate the method, we will work out the calculation in a few cases.

5.1 Strongly Regular

We begin with the formulas for $P(n, r, t)$ that follow from the the strongly regular graphs with an even number of vertices and no 3-cycles. These are trace-minimal.

Theorem 5.1

$$\begin{aligned}
P(19, 13, t) &= 20(5t + 2)^{10}(5t + 5)^9 \\
P(31, 21, t) &= 32(8t + 4)^{20}(8t + 8)^{11} \\
P(99, 29, t) &= 100(25t + 5)^{56}(25t + 10)^{42}(25t + 25) \\
P(111, 41, t) &= 112(28t + 8)^{70}(28t + 14)^{40}(28t + 28) \\
P(199, 89, t) &= 200(50t + 20)^{154}(50t + 30)^{44}(50t + 50).
\end{aligned}$$

The following table gives the graph class and a trace-minimal graph in the class corresponding to each pair (n, r). Since $r \equiv 1 \pmod 4$ in all cases, we use Theorem 1.2 and Equation (3) from Section 1.3. The characteristic polynomials for each graph is given in Section 4.3.

(n, r)	class	graph G or B
$(19, 13)$	$\mathcal{G}(10, 3)$	Petersen
$(31, 21)$	$\mathcal{G}(16, 5)$	Clebsh
$(99, 29)$	$\mathcal{G}(50, 7)$	Hoffman-Singleton
$(111, 41)$	$\mathcal{G}(56, 10)$	Gewirtz
$(199, 89)$	$\mathcal{G}(100, 22)$	Higman-Sims

As an example, take $(n, r) = (99, 29)$. Then $p = 25$ and $d = 7$. From Theorem 1.2, we seek a trace-minimal graph in $\mathcal{G}(2p, d)$, where $(2p, d) = (50, 7)$ The Hoffman-Singleton graph G is trace-minimal in $\mathcal{G}(50, 7)$ and has characteristic polynomial $\mathrm{ch}(G, x) = (x - 7)(x - 2)^{28}(x + 3)^{21}$. (See Section 4.3.) From Theorem 1.2, Equation (3) we have

$$\begin{aligned}
P(99, 29, t) &= \frac{4(t + 1)[\mathrm{ch}(G, pt + d)]^2}{t^2} \\
&= \frac{4(t + 1)[(25t)(25t + 5)^{28}(25t + 10)^{21}]^2}{t^2},
\end{aligned}$$

which is equal to the expression for $P(99, 29, t)$ given in Theorem 5.1.

5.2 Generalized polygons

In this section we give the formulas for $P(n, r, t)$ that correspond to the incidence graphs of generalized polygons, all of which are trace-minimal.

5.2.1 Finite projective planes

Theorem 5.2 *Let Γ be a finite projective plane of order q on $p = q^2 + q + 1$ points and lines, and let $n = 4p - 1$. Then*

$$P(n, 4q + 5, t) = 4p(pt + p)(pt + 2q + 2)^2((pt + q + 1)^2 - q)^{2p-2}.$$

In particular, for the finite projective plane of order $q = 2$ on seven points,

$$P(27, 13, t) = 28(7t + 7)(7t + 6)^2(49t^2 + 42t + 7)^{12},$$

which is one of the formulas in Theorem 5.23.

Let $d = q + 1$. The incidence graph $PP(q)$ of Γ is a trace-minimal d-regular graph in $\mathcal{G}(2p, q + 1)$ with $\mathrm{ch}(PP(q), x) = (x^2 - (q + 1)^2)(x^2 - q)^{p-1}$. (See Section 4.4.1.) Since $r = 4q + 5 \equiv 1 \pmod 4$, Theorem 5.2 follows from Theorem 1.2 as follows:

$$\begin{aligned}
P(n, 4q + 5, t) &= \frac{4(t + 1)[\mathrm{ch}(PP(q), pt + q + 1)]^2}{t^2} \\
&= \frac{4(t + 1)(pt)^2(pt + 2q + 2)^2[(pt + q + 1)^2 - q]^{2p-2}}{t^2} \\
&= 4p(pt + p)(pt + 2q + 2)^2((pt + q + 1)^2 - q)^{2p-2}.
\end{aligned}$$

5.2.2 Generalized quadrangles

Let q be a power of a prime, let Γ be the generalized quadrangle and $GQ(q)$ the incidence graph of Γ described in Section 4.4.2. Then $GQ(q)$ is a $(q + 1)$-regular bipartite graph on $v = 2(q^4 - 1)/(q - 1)$ vertices that is trace-minimal in $\mathcal{G}(v, q + 1)$ and hence bipartite-trace-minimal in $\mathcal{B}(v, q + 1)$. And from Lemma 4.7, the characteristic polynomial of $A(GQ(q))$ is

$$\mathrm{ch}(GQ(q), x) = x^{2a}(x^2 - 2q)^b(x^2 - (q + 1)^2),$$

where $a = q(q^2 + 1)/2$, and $b = q(q + 1)^2/2$. Then

$$\begin{aligned}
(pt + d)^2 - 2q &= (pt)^2 + 2(q + 1)pt + q^2 + 1, \\
(pt + d)^2 - (q + 1)^2 &= pt(pt + 2(q + 1)),
\end{aligned}$$

and so

$$\mathrm{ch}(GQ(q), pt + d) = (pt + q + 1)^{2a}((pt)^2 + 2(q + 1)pt + q^2 + 1)^b pt(pt + 2(q + 1)).$$

Since the generalized quadrilateral graph $GQ(q)$ is trace-minimal and bipartite-trace-minimal (Section 4.4.2), it can be used in conjunction with Theorems 1.2, 1.4, and 1.5 to obtain three formulas for $P(n, r, t)$.

The first formula comes from the fact that $GQ(q)$ is in $\mathcal{G}(2p, q+1)$, where $p = q^3+q^2+q+1 = (q+1)(q^2+1)$. We use Theorem 1.2 with $r = 4d + 1 = 4q + 5$. Thus

$$\begin{aligned}
&\frac{4(t + 1)[\mathrm{ch}(GQ(q), pt + d)]^2}{t^2} \\
&= 4p(pt + p)(pt + q + 1)^{2q(q^2+1)} \\
&\quad \times((pt)^2 + 2(q + 1)pt + q^2 + 1)^{q(q+1)^2}(pt + 2(q + 1))^2,
\end{aligned}$$

which proves the next theorem.

Theorem 5.3 *Let q be a power of a prime, and $p=(q+1)(q^2+1)$ and let $n=4p-1$. Then*

$$\begin{aligned}&P(n,4q+5,t)\\ &= 4p(pt+p))(pt+q+1)^{2q(q^2+1)}((pt)^2+2(q+1)pt+q^2+1)^{q(q+1)^2}(pt+2(q+1))^2.\end{aligned}$$

Our next formula comes from the fact that $GQ(q)\in\mathcal{B}(4p,q+1)$, where $p=(q+1)(q^2+1)/2$. (Since q is odd, p is an integer.) Let $r=4d-1=4q+3$. Then $d<p/2$, $p(t-1)+2d=pt-(q+1)(q^2+1)/2$, and

$$\begin{aligned}&\frac{4(p(t-1)+2d)\mathrm{ch}(GQ(q),pt+d)}{t(pt+2d)}\\ &= 4p(pt-(q+1)(q^2-3)/2)(pt+q+1)^{q(q^2+1)}\\ &\quad\times((pt)^2+2(q+1)pt+q^2+1)^{q(q+1)^2/2}.\end{aligned}$$

Using Theorem 1.4, we have proved the next theorem.

Theorem 5.4 *Let q be a power of an odd prime, $p=(q+1)(q^2+1)/2$, and $n=4p-1$. Then*

$$\begin{aligned}&P(n,4q+3,t)\\ &= 4p(pt-(q+1)(q^2-3)/2)(pt+q+1)^{q(q^2+1)}((pt)^2+2(q+1)pt+q^2+1)^{q(q+1)^2/2}.\end{aligned}$$

Finally, Theorem 1.5 applies to the incidence graph $GQ(q)\in\mathcal{G}(4p,q+1)$, where $r=4d$, and $p=(q+1)(q^2+1)/2$. Then $d\le p/2$ and

$$\begin{aligned}&\frac{4\mathrm{ch}(GQ(q),(pt+d))}{t}\\ &= 4p(pt+q+1)^{q(q^2+1)}(pt+2(q+1))[(pt)^2+2(q+1)pt+q^2+1]^{q(q+1)^2/2}.\end{aligned}$$

Thus we have the next theorem.

Theorem 5.5 *Let q be a power of an odd prime, $p=(q+1)(q^2+1)/2$, and $n=4p-1$. Then*

$$P(n,4q+4,t) = 4p(pt+q+1)^{q(q^2+1)}(pt+2(q+1))[(pt)^2+2(q+1)pt+q^2+1]^{q(q+1)^2/2}.$$

5.2.3 Generalized hexagons

Let q be a power of a prime, let Γ be the generalized hexagon and $GH(q)$ the incidence graph of Γ described in Section 4.4.3. Then $GH(q)$ is a $(q+1)$-regular bipartite graph on $v=2(q^6-1)/(q-1)$ vertices that is trace-minimal in $\mathcal{G}(v,q+1)$ and hence bipartite-trace-minimal in $\mathcal{B}(v,q+1)$. The characteristic polynomial of $A(GH(q))$ is

$$\mathrm{ch}(GH(q),x)=x^{2a}\left(x^2-q\right)^b\left(x^2-3q\right)^c\left(x^2-(q+1)^2\right)$$

where $a=q(q^2+q+1)(q^2-q+1)/3$, $b=q(q^2-q+1)\left(q+1\right)^2/2$, and $c=q\left(q+1\right)^2(q^2+q+1)/6$. Throughout this section $d=q+1$. Thus

$$\begin{aligned}(pt+d)^2-q &= (pt)^2+2\left(q+1\right)pt+q^2+q+1,\\ (pt+d)^2-3q &= (pt)^2+2\left(q+1\right)pt+q^2-q+1,\\ (pt+d)^2-(q+1)^2 &= pt\left(pt+2(q+1)\right),\end{aligned}$$

and so

$$\begin{aligned}\mathrm{ch}(GH(q), pt+d) &= (pt+q+1)^{2a}\left((pt)^2+2\,(q+1)\,pt+q^2+q+1\right)^b \\ &\quad \left((pt)^2+2\,(q+1)\,pt+q^2-q+1\right)^c pt\,(pt+2(q+1))\,.\end{aligned}$$

Since $GH(q)$ is trace-minimal and bipartite-trace-minimal, it can be used in conjunction with Theorems 1.2, 1.4, and 1.5 to obtain three formulas for $P(n,r,t)$.

First, $GH(q)$ is in $\mathcal{G}(2p,q+1)$ where $p=(q^6-1)/(q-1)$. We use Theorem 1.2 with $r=4d+1=4q+5$. We have

$$\begin{aligned}\frac{4(t+1)\left[\mathrm{ch}(GH(q),pt+d)\right]^2}{t^2} &= 4p^2(t+1)\,(pt+2(q+1))^2\,(pt+q+1)^{4a} \\ &\quad \times\left((pt)^2+2\,(q+1)\,pt+q^2+q+1\right)^{2b}\left((pt)^2+2\,(q+1)\,pt+q^2-q+1\right)^{2c}.\end{aligned}$$

Thus we have the next theorem.

Theorem 5.6 *Let q be a power of a prime, and $p=(q^6-1)/(q-1)$ and let $n=4p-1$. Then*

$$\begin{aligned}P(n,4q+5,t) &= 4p^2(t+1)\,(pt+2(q+1))^2\,(pt+q+1)^{4a} \\ &\quad \times\left((pt)^2+2\,(q+1)\,pt+q^2+q+1\right)^{2b}\left((pt)^2+2\,(q+1)\,pt+q^2-q+1\right)^{2c}.\end{aligned}$$

Next, let $r=4q+3=4d-1$. In this case we use Theorem 1.4 with $GH(q)\in\mathcal{B}(4p,q+1)$, where $p=(q^6-1)/2(q-1)$. (Since q is odd, p is an integer.) Then $d<p/2$, $p(t-1)+2d=pt-(q+1)(q^4+q^2-3)/2$, and

$$\begin{aligned}&\frac{4(p(t-1)+2d)ch(G,pt+d)}{t(pt+2d)} = 4p(pt-(q+1)(q^4+q^2-3)/2)(pt+q+1)^{2a} \\ &\quad \times\left((pt)^2+2\,(q+1)\,pt+q^2+q+1\right)^b\left((pt)^2+2\,(q+1)\,pt+q^2-q+1\right)^c.\end{aligned}$$

Thus we have the next theorem.

Theorem 5.7 *Let q be a power of an odd prime, and $p=(q^6-1)/2(q-1)$ and let $n=4p-1$. Then*

$$\begin{aligned}P(n,4q+3,t) &= 4p(pt-(q+1)(q^4+q^2-3)/2)(pt+q+1)^{2a} \\ &\quad \times\left((pt)^2+2\,(q+1)\,pt+q^2+q+1\right)^b\left((pt)^2+2\,(q+1)\,pt+q^2-q+1\right)^c.\end{aligned}$$

Finally, let $r=4d=4q+4$. In this case we use Theorem 1.5 with $GH(q)\in\mathcal{G}(4p,q+1)$, where $p=(q^6-1)/2(q-1)$. Then $d\le p/2$ and

$$\begin{aligned}&\frac{4ch(G,pt+d)}{t} = 4p(pt+2(q+1))\,(pt+q+1)^{2a} \\ &\quad \times\left((pt)^2+2\,(q+1)\,pt+q^2+q+1\right)^b\left((pt)^2+2\,(q+1)\,pt+q^2-q+1\right)^c.\end{aligned}$$

Thus we have the next theorem.

Theorem 5.8 *Let q be a power of an odd prime, and $p = (q^6 - 1)/2(q - 1)$ and let $n = 4p - 1$. Then*

$$\begin{aligned} P(n, 4q+4, t) &= 4p(pt + 2(q+1))\,(pt + q + 1)^{2a} \\ &\quad \times \left((pt)^2 + 2\,(q+1)\,pt + q^2 + q + 1\right)^b \left((pt)^2 + 2\,(q+1)\,pt + q^2 - q + 1\right)^c. \end{aligned}$$

5.3 Large r

In this section we give formulas for $P(n, r, t)$ for all pairs (n, r) satisfying $n - 21 \le r \le n - 1$ except $r = n - 10, n - 11, n - 14, n - 15$. These exceptional values of r require (bipartite-) trace-minimal graphs in $\mathcal{G}(2p, p-3)$, $\mathcal{B}(4p, 2p-3)$, $\mathcal{G}(2p, p-4)$, and $\mathcal{B}(4p, 2p-4)$. We have not analyzed these cases.

We assume throughout that p is a positive integer and $n = 4p-1$. Some of the formulas in this section do not hold for some small values of n. For example, the formula in the first part of Theorem 5.10 does not hold for $n = 15, 19$. We will not, however, restate the formulas for $P(n, r, t)$ for $n = 3, 7, 11, 15, 19, 23, 27$ as they appear either in [AFNW] or in Section 5.5, 5.6, or 5.7.

5.3.1 $r = n - 1, n - 3, n - 4, n - 5$

Theorem 5.9

$$\begin{aligned} P(n, n-1, t) &= 4p(pt)(pt + p)^{4p-2} \\ P(n, n-3, t) &= 4p(pt + 2p - 2)(pt + p - 2)^{2p-1}(pt + p)^{2p-1}, \quad \textit{for } n \ge 11 \\ P(n, n-4, t) &= 4p(pt + p)^{2p-1}(pt + p - 2)^{2p}, \quad \textit{for } n \ge 7 \\ P(n, n-5, t) &= 4p(pt)(pt + p)^{2p-2}(pt + p - 2)^{2p}, \quad \textit{for } n \ge 7. \end{aligned}$$

The graphs that are required to prove the cases $r = n-1$, $n-3$, $n-4$, and $n-5$ of Theorem 5.9 appear in Section 4.1 and are given in the following table along with the appropriate theorem and equation, which depend on $r \pmod 4$:

r	class	graph G	Theorem/Equation
$n-1$	$\mathcal{G}(2p, 2p-1)$	K_{2p}	1.3 (4)
$n-3$	$\mathcal{B}(4p, 2p-1)$	$K_{2p,2p} - 2pK_2$	1.5 (8)
$n-4$	$\mathcal{G}(4p, 4p-2)$	$I_2^{(2p)}$	1.4 (6)
$n-5$	$\mathcal{G}(2p, 2p-2)$	$I_2^{(p)}$	1.3 (4)

5.3.2 $r = n - 2$

Theorem 5.10 *If $n \ne 15, 19$, then*

$$P(n, n-2, t) = 4p(pt + p)^{2p-1}(pt + 2p - 2)^2(pt + p - 2)^{2p-2}.$$

Let $r = n - 2 = 4p - 3 \equiv 1 \pmod 4$. Then Theorem 1.2 applies with $d = p - 1$. We seek a trace-minimal graph in $\mathcal{G}(2p, p-1)$. If $p \ne 4, 5$, then $K_{p,p} - pK_2$ is trace minimal. See Section 4.7.2. For $p = 4, 5$ the sporadic graph $S(8,3)$ is trace-minimal in $\mathcal{G}(8,3)$ and $S(10,4)$ is trace-minimal in $\mathcal{G}(10,4)$. See Section 4.6. Theorem 5.10 now follows from Theorem 1.2.

5.3.3 $r = n-6$

Theorem 5.11 *If $n \geq 43$ or $n = 35$, then*

$$P(n,n-6,t) = 4p\frac{(pt+2p-1)^2}{(pt+p)(pt+p-4)^2}\left[2\mathrm{Tch}_{2p}\left(\frac{pt+p-2}{2}\right)-2\right]^2. \tag{19}$$

For $n = 31, 39$ we have

$$\begin{aligned} P(31,25,t) &= 32(8t+4)^4(8t+6)^{16}(8t+8)^3((8t)^2+16(8t)+56)^4 \\ P(39,33,t) &= 40(10t+8)^{30}(10t+10)((10t)^2+20(10t)+80)^4. \end{aligned}$$

For $r = n-6 = 4p-7 \equiv 1 \pmod 4$. Theorem 1.2 applies with $d = p-2$. If $p \geq 11$ or $p = 9$ ($n \geq 43$ or $n = 35$), then $G = K_{p,p} - C_{2p}$ is a trace-minimal graph in $\mathcal{G}(2p, p-2)$. (See Section 4.7.2.)

For $p = 8, 10$, the sporadic graph $S(16,6)$ is trace-minimal in $\mathcal{G}(16,6)$ and $S(20,8)$ is trace-minimal in $\mathcal{G}(20,8)$. (See Section 4.6.)

5.3.4 $r = n-7$

Theorem 5.12 *If $n \geq 19$, then*

$$P(n,n-7,t) = 4p\frac{pt+2p-4}{(pt+p)(pt+p-4)}\left[2\mathrm{Tch}_{4p}\left(\frac{pt+p-2}{2}\right)-2\right].$$

For $r = n-7 = 4p-8 \equiv 0 \pmod 4$), thus we use Theorem 1.5 with $d = p-2$. Suppose $p \geq 5$, that is $n \geq 19$. Then $p/2 < d$ and so Equation (8) of Theorem 1.5 applies. We seek a bipartite-trace-minimal graph B in $\mathcal{B}(4p, 2p-2)$. From Section 4.2, $B = K_{2p,2p} - C_{4p}$.

5.3.5 $r = n-8$, $n-9$

These formulas depend on the congruence class of $n \pmod{12}$. Let k be a positive integer.

Theorem 5.13 *The polynomials $P(n,n-8,t)$ and $P(n,n-9,t)$ are given in the following tables:*

n	p	$P(n,n-8,t)$
$12k-1 \geq 23$	$3k$	$4p(pt+p-3)^{8k}(pt+p)^{4k-1}$
$12k+3 \geq 15$	$3k+1$	$4p(pt+p-3)^{8k}(pt+p)^{4k}(pt+p-2)^2(pt+p-4)$
$12k+7 \geq 19$	$3k+2$	$4p(pt+p-3)^{8k+2}(pt+p)^{4k+1}((pt)^2+(2p-5)(pt)+p^2-5p+5)^2$.

n	p	$P(n, n-9, t)$
$12k-1 \geq 11$	$3k$	$4p(pt)(pt+p-3)^{8k}(pt+p)^{4k-2}$
$12k+3 \geq 15$	$3k+1$	$4p(pt)(pt+p-3)^{8k-4}(pt+p)^{4k-2}((pt)^2+(2p-5)(pt)+p^2-5p+5)^4$
$12k+7 \geq 19$	$3k+2$	$4p(pt)(pt+p-3)^{8k}(pt+p)^{4k}(pt+p-2)^4(pt+p-4)^2$.

Let $r = n-8 = 4p-9 \equiv -1 \pmod 4$. Thus we use Theorem 1.4 with $d = p-2$. Since $p/2 \leq d$ for $p \geq 4$, Equation (5) of Theorem 1.4 applies. We seek a trace-minimal graph in $\mathcal{G}(4p, 4p-3)$. The description depends on the congruence class of $v \pmod 3$. Since $v = 4p$ we must consider the cases $v \equiv 0, 4, 8 \pmod{12}$. Lemma 4.16 describes the trace-minimal graphs in $\mathcal{G}(v, v-3)$, which we describe in the following table:

n	p	graph class	graph
$12k-1 \geq 23$	$3k$	$\mathcal{G}(12k, 12k-3)$	$I_3^{(4k)}$
$12k+3 \geq 15$	$3k+1$	$\mathcal{G}(12k+4, 12k+1)$	$2K_2 \bigtriangledown I_3^{(4k)}$
$12k+7 \geq 19$	$3k+2$	$\mathcal{G}(12k+8, 12k+5)$	$C_5 \bigtriangledown I_3^{(4k+1)}$

Suppose $n = 12k-1$. Then $p = 3k$, $d = 3k-2$, and $r = 12k-9$. Taking $l = 4k$ in Lemma 4.16, we have that $G = I_3^{(4k)}$ is the unique trace-minimal graph in $\mathcal{G}(12k, 12k-3)$. And $\mathrm{ch}(G, x) = (x-(12k-3))x^{8k}(x+3)^{4k-1}$. Thus from Theorem 1.4 we have

$$\begin{aligned} P(12k-1, 12k-9, t) &= \frac{4\mathrm{ch}(G, pt+3k-3)}{t-3} \\ &= 4p(pt+3k)^{4k-1}(pt+3k-3)^{8k}. \end{aligned}$$

The first part of Theorem 5.13 is proved. The two other parts of the case $r = n-8$ are proved in a similar way.

Now let $r = n-9 = 4p-10 \equiv 2 \pmod 4$. We use Theorem 1.3 with $d = p-3$. We seek a trace-minimal graph in $\mathcal{G}(2p, 2p-3)$. Lemma 4.16 describes the trace-minimal graphs in $\mathcal{G}(v, v-3)$. As in the case of $r = n-8$ the description depends on the congruence class of $v \pmod 3$ and since $v = 2p$ we must consider the cases $v \equiv 0, 2, 4 \pmod 6$. These cases correspond to $n \equiv -1, 3, 7 \pmod{12}$ since $n = 4p-1$.

The graphs and their characteristic polynomials that are required to prove the case $r = n-9$ are given in the following table:

n	p	graph class	graph
$12k-1 \geq 11$	$3k$	$\mathcal{G}(6k, 6k-3)$	$I_3^{(2k)}$
$12k+3 \geq 15$	$3k+1$	$\mathcal{G}(6k+2, 6k-1)$	$C_5 \bigtriangledown I_3^{(2k-1)}$
$12k+7 \geq 19$	$3k+2$	$\mathcal{G}(6k+4, 6k+1)$	$2K_2 \bigtriangledown I_3^{(2k)}$

The remaining parts of Theorem 5.13 are established with the same type of arguments used in the case $r = n-8$.

For example, suppose $n = 12k+3$. Then $p = 3k+1$, $r = 12k-6$, and $d = 3k-2$. The trace-minimal graph in $\mathcal{G}(6k+2, 6k-1)$ is $G = C_5 \bigtriangledown I_3^{(2k-1)}$ with $\mathrm{ch}(G, x) = (x-(6k-1))x^{4k-2}(x+3)^{2k-1}(x^2+x-1)^2$.

Hence

$$\begin{aligned} P(12k+3, 12k-6, t) &= \frac{4t[\mathrm{ch}(G, pt+3k-2)]^2}{(t-1)^2} \\ &= \frac{4t[(pt-p)(pt+3k-2)^{4k-2}(pt+3k+1)^{2k-1}((pt)^2+(6k-3)pt+9k^2-9k+1)^2]^2}{(t-1)^2} \\ &= 4p(pt)(pt+3k-2)^{8k-4}(pt+3k+1)^{4k-2}((pt)^2+(6k-3)pt+9k^2-9k+1)^4. \end{aligned}$$

5.3.6 $r = n-12$, $n-13$

Theorem 5.14 *Let $n \geq 23$. Then*

$$P(n, n-12, t) = 4p(pt+p-4)^{3p}(pt+p)^{p-1}.$$

The formulas for $r = n-13$ depend on the congruence class of $n \pmod 8$.

Theorem 5.15 *Let k be a positive integer. The polynomials $P(n, n-13, t)$ are given in the following table:*

n	p	$P(n, n-13, t)$
$8k-1 \geq 15$	$2k$	$4p(pt)(pt+p-4)^{6k}(pt+p)^{2k-2}$.
$8k+3 \geq 19$	$2k+1$	$4p(pt)(pt+p-4)^{6k-6}(pt+p-5)^4(pt+p-3)^4(pt+p-2)^2(pt+p)^{2k-2}$.

Let $r = n-12 = 4p-13 \equiv -1 \pmod 4$. Theorem 1.4 applies with $d = p-3$. For $p/2 \leq d$, ($p \geq 6$, $n \geq 23$) we seek a trace-minimal graph in $\mathcal{G}(4p, 4p-4)$, which from Section 4.8.3 is $I_4^{(p)}$. Then Theorem 5.14 follows.

Now let $r = n-13 = 4p-14 \equiv 2 \pmod 4$. Theorem 1.3 applies with $d = p-4$. We seek a trace-minimal graphS in $\mathcal{G}(2p, 2p-4)$, which according to Section 4.8.3 are given in the following table:

n	p	graph class	graph
$8k-1$	$2k$	$\mathcal{G}(4k, 4k-4)$	$I_4^{(k)}$
$8k+3$	$2k+1$	$\mathcal{G}(4k+2, 4k-2)$	$C_6 \bigtriangledown I_4^{(k-1)}$.

5.3.7 $r = n-16, n-17$

These formulas depend on the congruence class of $n \pmod{20}$.

Theorem 5.16 *Let k be a positive integer. The polynomials $P(n, n-16, t)$ are given in the following table:*

n	p	$P(n,n-16,t)$
$20k-1 \geq 39$	$5k$	$4p(pt+p-5)^{16k}(pt+p)^{4k-1}$
$20k+3 \geq 43$	$5k+1$	$4p(pt+p-6)^2(pt+p-5)^{16k-2}(pt+p-4)(pt+p-3)$
		$\times(pt+p)^{4k-1}((pt)^2+(2p-7)(pt)+(p^2-7p+8))$
$20k+7 \geq 47$	$5k+2$	$4p(pt+p-6)^2(pt+p-5)^{16k}(pt+p-4)(pt+p)^{4k}$
		$\times((pt)^2+(2p-8)(pt)+(p^2-8p+14))^2$
$20k+11 \geq 31$	$5k+3$	$4p(pt+p-5)^{16k+4}(pt+p)^{4k+1}((pt)^3+(3p-14)(pt)^2+$
		$+(3p^2-28p+63)(pt)+(p^3-14p^2+63p-91))^2$
$20k+15 \geq 35$	$5k+4$	$4p(pt+p-6)^2(pt+p-5)^{16k+8}(pt+p-4)^3(pt+p)^{4k+2}.$

Theorem 5.17 *Let k be a positive integer. The polynomials $P(n,n-17,t)$ are given in the following table:*

n	p	$P(n,n-17,t)$
$20k-1 \geq 19$	$5k$	$4p(pt)(pt+p-5)^{16k}(pt+p)^{4k-2}$
$20k+3 \geq 23$	$5k+1$	$4p(pt)(pt+p-5)^{16k-8}(pt+p)^{4k-2}((pt)^3+(3p-14)(pt)^2+$
		$+(3p^2-28p+63)(pt)+(p^3-14p^2+63p-91))^4$
$20k+7 \geq 27$	$5k+2$	$4p(pt)(pt+p-6)^4(pt+p-5)^{16k-4}(pt+p-4)^2(pt+p-3)^2$
		$\times(pt+p)^{4k-2}((pt)^2+(2p-7)(pt)+(p^2-7p+8))^2$
$20k+11 \geq 31$	$5k+3$	$4p(pt)(pt+p-6)^4(pt+p-5)^{16k}(pt+p-4)^6(pt+p)^{4k}$
$20k+15 \geq 35$	$5k+4$	$4p(pt)(pt+p-6)^4(pt+p-5)^{16k}(pt+p-4)^2(pt+p)^{4k}$
		$\times((pt)^2+(2p-8)(pt)+(p^2-8p+14))^4$

Let $r = n-16$. Then $r = 4p-17 \equiv -1 \pmod 4$. Thus we use Theorem 1.4 with $d = p-4$. Suppose $p \geq 8$ so that $p/2 \leq d$ and Equation (5) of Theorem 1.4 applies. We seek a trace-minimal graph in $\mathcal{G}(4p, 4p-5)$. Trace- minimal graphs in $\mathcal{G}(v, v-5)$ are given in Section 4.18. The graphs depend on the congruence class of $v \pmod 5$ and since $v = 4p$ we must consider the cases $v \equiv 0, 4, 8, 12, 16 \pmod{20}$, which correspond to the cases $n \equiv -1, 3, 7, 11, 15 \pmod{20}$. The graphs that are required to prove the case $r = n-16$ are given in the following table:

n	p	graph class	graph
$20k-1 \geq 39$	$5k$	$\mathcal{G}(20k, 20k-5)$	$I_5^{(4k)}$
$20k+3 \geq 43$	$5k+1$	$\mathcal{G}(20k+4, 20k-1)$	$S(9,4) \bigtriangledown I_5^{(4k-1)}$
$20k+7 \geq 47$	$5k+2$	$\mathcal{G}(20k+8, 20k+3)$	$S(8,3) \bigtriangledown I_5^{(4k)}$
$20k+11 \geq 31$	$5k+3$	$\mathcal{G}(20k+12, 20k+7)$	$C_7 \bigtriangledown I_5^{(4k+1)}$
$20k+15 \geq 35$	$5k+4$	$\mathcal{G}(20k+16, 20k+11)$	$3K_2 \bigtriangledown I_5^{(4k+2)}$

The trace-minimal graphs and characteristic polynomials for these graph classes are given in Section 4.8.4.

To prove Theorem 5.16, we use Theorem 1.4 and calculate $4\mathrm{ch}(G,x)/(t-3)$ with $x = pt+d-1 = pt+p-5$. For example, if $n = 20k+11$ and $p = 5k+3$, set $x = pt+5k-2$. Then

$$P(20k+11, 20k-5, t) = \frac{4\mathrm{ch}(C_7 \bigtriangledown I_5^{(4k+1)}, pt+5k-2)}{t-3},$$

which results in the polynomial in Theorem 5.16. Notice that if $x = pt+5k-2$, then $x-(20k+7) = p(t-3)$. The other parts of Theorem5.16 are proved in a similar way.

Let $r = n-17 = 4p-18 \equiv 2 \pmod 4$. We use Theorem 1.3 with $d = p-5$. We seek trace-minimal graphs in $\mathcal{G}(2p, 2p-5)$, which are given in Section 4.8.4. Thus we consider the congruence class of $n = 4p-1 \pmod{20}$. The graphs that are required to prove the case $r = n-17$ are given in the following table:

n	p	graph class	graph
$20k-1 \geq 19$	$5k$	$\mathcal{G}(10k, 10k-5)$	$I_5^{(2k)}$
$20k+3 \geq 23$	$5k+1$	$\mathcal{G}(10k+2, 10k-3)$	$C_7 \bigtriangledown I_5^{(2k-1)}$
$20k+7 \geq 27$	$5k+2$	$\mathcal{G}(10k+4, 10k-1)$	$S(9,4) \bigtriangledown I_5^{(2k-1)}$
$20k+11 \geq 31$	$5k+3$	$\mathcal{G}(10k+6, 10k+1)$	$3K_2 \bigtriangledown I_5^{(2k)}$
$20k+15 \geq 35$	$5k+4$	$\mathcal{G}(10k+8, 10k+3)$	$S(8,3) \bigtriangledown I_5^{(2k)}$

To prove Theorem 5.17, we use Theorem 1.3 and calculate $4t[\mathrm{ch}(G,x)]^2/(t-1)^2$ with $x = pt+d = pt+p-5$. For example, if $n = 20k+11$ and $p = 5k+3$, set $x = pt+5k-2$. Then

$$P(20k+11, 20k-6, t) = \frac{4t[\mathrm{ch}(3K_2 \bigtriangledown I_5^{(2k)}, pt+5k-2)]^2}{(t-1)^2},$$

which results in the polynomial in Theorem 5.17. The other parts of Theorem 5.17 are proved in a similar way.

Theorem 5.18 *Let k be a positive integer. The polynomials $P(n, n-20, t)$ are given in the following table:*

n	p	*P(n,n-20,t)*
$12k-1 \geq 47$	$3k$	$4p(pt+p-6)^{10k}(pt+p)^{2k-1}$
$12k+3 \geq 39$	$3k+1$	$4p(pt+p-6)^{10k}(pt+p)^{2k-1}$
		$\times((pt)^2+2(p-5)(pt)+(p^2-10p+20))^2$
$12k+7 \geq 43$	$3k+2$	$4p(pt+p-6)^{10k+2}(pt+p-4)(pt+p)^{2k}$
		$\times((pt)^2+2(p-6)(pt)+(p^2-12p+34))^2$

Theorem 5.19 *Let k be a positive integer. The polynomials $P(n, n-21, t)$ are given in the following table:*

n	p	*P(n,n-21,t)*
$12k-1 \geq 23$	$3k$	$4p(pt)(pt+p-6)^{10k}(pt+p)^{2k-2}$
$12k+3 \geq 27$	$3k+1$	$4p(pt)(pt+p-6)^{10k-6}(pt+p-4)^2(pt+p)^{2k-2}$
		$\times((pt)^2+2(p-6)(pt)+(p^2-12p+34))^4$
$12k+7 \geq 31$	$3k+2$	$4p(pt)(pt+p-6)^{10k}(pt+p)^{2k-2}$
		$\times((pt)^2+2(p-5)(pt)+(p^2-10p+20))^4$

Let $r = n-20$. Then $r = 4p-21 \equiv -1 \pmod 4$. Thus we use Theorem 1.4 with $d = p-5$. Suppose $p \geq 10$ so that $p/2 \leq d$ and Equation (5) of Theorem 1.4 applies. We seek a trace-minimal graph in

$\mathcal{G}(4p, 4p-6)$. Trace minimal graphs in $\mathcal{G}(v, v-6)$ are given in Section 6.8.6. The graphs depend on the congruence class of $v \pmod 6$ and since $v = 4p$ we must consider $v \equiv 0, 4, 8 \pmod{12}$, which correspond to the cases $n \equiv -1, 3, 7 \pmod{12}$. The graphs that are required to prove the case $r = n - 20$ are given in the following table:

n	p	graph class	graph
$12k-1 \geq 47$	$3k$	$\mathcal{G}(12k, 12k-6)$	$I_6^{(2k)}$
$12k+3 \geq 39$	$3k+1$	$\mathcal{G}(12k+4, 12k-2)$	$S(10,4) \bigtriangledown I_6^{(2k-1)}$
$12k+7 \geq 43$	$3k+2$	$\mathcal{G}(12k+8, 12k+2)$	$C_8 \bigtriangledown I_6^{(2k)}$

To prove Theorem 5.18 we use Theorem 1.4 and calculate $4\mathrm{ch}(G,x)/(t-3)$ with $x = pt+d-1 = pt+p-6$. For example, if $n = 12k+3$ and $p = 3k+1$, set $x = pt + 3k - 5$. Then

$$P(12k+3, 12k-17, t) \quad = \quad \frac{4\mathrm{ch}(S(10,4) \bigtriangledown I_6^{(2k-1)}, pt+3k-3)}{t-3},$$

which results in the polynomial in Theorem 5.18. The other parts of Theorem 5.18 are proved in a similar way.

Let $r = n - 21 = 4p - 22 \equiv 2 \pmod 4$. We use Theorem 1.3 with $d = p - 6$. Note that this implies $p \geq 6$ as $d \geq 0$. We seek trace-minimal graphs in $\mathcal{G}(2p, 2p-6)$, which are given in Section 4.8.5. Thus we consider the congruence class of $n = 4p-1 \pmod{12}$. The graphs that are required to prove the case $r = n - 21$ are given in the following table:

n	p	graph class	graph
$12k-1 \geq 23$	$3k$	$\mathcal{G}(6k, 6k-6)$	$I_6^{(k)}$
$12k+3 \geq 27$	$3k+1$	$\mathcal{G}(6k+2, 6k-4)$	$C_8 \bigtriangledown I_6^{(k-1)}$
$12k+7 \geq 31$	$3k+2$	$\mathcal{G}(6k+4, 6k-2)$	$S(10,4) \bigtriangledown I_6^{(k-1)}$

To prove Theorem 5.19 we use Theorem refthm:mainr2 and calculate $4t[\mathrm{ch}(G,x)]^2/(t-1)^2$ with $x = pt + d = pt + p - 6$. For example, if $n = 12k+3$ and $p = 3k+1$, set $x = pt + 3k - 5$. Then

$$P(12k+3, 12k-18, t) \quad = \quad \frac{4t[\mathrm{ch}(C_8 \bigtriangledown I_6^{(k-1)}, pt+3k-5)]^2}{(t-1)^2},$$

which results in the polynomial in Theorem 5.19. The other parts of Theorem 5.19 are proved in a similar way.

5.4 Small r

Theorem 5.20 *The polynomials $P(n,r,t)$ for $0 \leq r \leq 9$ but $r \neq 6$ are as follows:*

$$
\begin{aligned}
P(n,0,t) &= 4p(pt)^{4p-1}\\
P(n,1,t) &= 4p(pt)^{4p-2}(pt+p)\\
P(n,2,t) &= 4p(pt)^{4p-3}(pt+p)^2,\ \ n\geq 7\\
P(n,3,t) &= 4p(pt)^{2p-1}(pt+2)^{2p-1}(pt-p+2),\ \ n\geq 11\\
P(n,4,t) &= 4p(pt)^{2p-1}(pt+2)^{2p}\\
P(n,5,t) &= 4p(pt)^{2p-2}(pt+2)^{2p}(pt+p)\\
P(n,7,t) &= \frac{4[2\mathrm{Tch}_{4p}((pt+2)/2)-2]}{t},\ \ n\geq 19\\
P(n,8,t) &= \frac{4(pt+4)[2\mathrm{Tch}_{4p}((pt+2)/2)-2]}{(t-1)(pt+p+4)},\ \ n\geq 15\\
P(n,9,t) &= \frac{4(t+1)[2\mathrm{Tch}_{2p}((pt+2)/2)-2]^2}{t^2}.
\end{aligned}
$$

We are not able to give a formula for $P(n,6,t)$ at this time.

The graphs and their characteristic polynomials that are required to prove Theorem 5.20 are given in the following table:

r	graph class	graph	characteristic polynomial
0	$\mathcal{G}(4p,0)$	I_{4p}	x^{4p}
1	$\mathcal{G}(2p,0)$	I_{2p}	x^{2p}
2	$\mathcal{G}(2p,p)$	$K_{p,p}$	$(x-p)(x+p)x^{2p-2}$
3	$\mathcal{B}(4p,1)$	$2pK_2$	$(x-1)^{2p}(x+1)^{2p}$
4	$\mathcal{G}(4p,1)$	$2pK_2$	$(x-1)^{2p}(x+1)^{2p}$
5	$\mathcal{G}(2p,1)$	pK_2	$(x-1)^{p}(x+1)^{p}$
7	$\mathcal{B}(4p,2)$	C_{4p}	$2\mathrm{Tch}_{4p}(x/2)-2$
8	$\mathcal{G}(4p,2)$	C_{4p}	$2\mathrm{Tch}_{4p}(x/2)-2$
9	$\mathcal{G}(2p,2)$	C_{2p}	$2\mathrm{Tch}_{2p}(x/2)-2.$

The proof that each graph is trace-minimal or bipartite-trace-minimal in its class is given in 4.1, for $r=0,1,3,4,5$, in 4.7.1 for $r=2$, and in 4.2 for $r=7,8,9$.

The proof of the case $r=2$ of Theorem 5.20, for example, falls under purview of Theorem 1.3 with $d=0$. The complete bipartite graph $G=K_{p,p}\in\mathcal{G}(2p,p)$ is trace-minimal and $A(K_{p,p})$ has characteristic polynomial $(x-p)(x+p)x^{2p-2}$. Thus

$$
\begin{aligned}
P(n,2,t) &= \frac{4t[\mathrm{ch}(G,pt)]^2}{(t-1)^2}\\
&= \frac{4t[(pt-p)(pt+p)(pt)^{2p-2}]^2}{(t-1)^2}\\
&= 4p(pt)^{4p-3}(pt+p)^2.
\end{aligned}
$$

The proofs of the other cases in Theorem 5.20 are similar.

Now consider the case $r = 6$. From Theorem 1.3 with $d = 1$, we need to find a trace-minimal graph in $\mathcal{G}(2p, p+1)$. We have not been able to analyze this case successfully.

5.5 $n = 19$

In this section we exhibit the polynomials $P(19, r, t)$ for $r = 0, 1, 2, \ldots, 18$. The section that describes the corresponding trace-minimal graph is also given here.

Theorem 5.21

$$\begin{aligned}
P(19,0,t) &= 20(5t)^{19} \\
P(19,1,t) &= 20(5t)^{18}(5t+5) \\
P(19,2,t) &= 20(5t)^{17}(5t+5)^2 \\
P(19,3,t) &= 20(5t-3)(5t)^9(5t+2)^9 \\
P(19,4,t) &= 20(5t)^9(5t+2)^{10} \\
P(19,5,t) &= 20(5t)^8(5t+2)^{10}(5t+5) \\
P(19,6,t) &= 20(5t)^5(5t+1)^6(5t+2)^4(5t+3)^2(5t+5)^2 \\
P(19,7,t) &= 20(5t+2)^2(5t-1)((5t)^2+3(5t)+1)^2((5t)^2+5(5t)+5)^2((5t)^4+8(5t)^3+19(5t)^2+12(5t)+1)^2 \\
P(19,8,t) &= 20(5t+2)^2(5t+4)((5t)^2+3(5t)+1)^2((5t)^2+5(5t)+5)^2((5t)^4+8(5t)^3+19(5t)^2+12(5t)+1)^2 \\
P(19,9,t) &= 20(5t+4)^2(5t+5)((5t)^2+3(5t)+1)^4((5t)^2+5(5t)+5)^4 \\
P(19,10,t) &= 20(5t)(5t+1)^2(5t+2)^8(5t+3)^4(5t+5)^4 \\
P(19,11,t) &= 20(5t+2)^{10}(5t+5)^5((5t)^2+5(5t)+5)^2 \\
P(19,12,t) &= 20(5t+3)^2(5t+6)((5t)^2+5(5t)+5)^2((5t)^2+7(5t)+11)^2((5t)^4+12(5t)^3+49(5t)^2+78(5t)+41)^2 \\
P(19,13,t) &= 20(5t+2)^{10}(5t+5)^9 \\
P(19,14,t) &= 20(5t)(5t+5)^8(5t+3)^{10} \\
P(19,15,t) &= 20(5t+5)^9(5t+3)^{10} \\
P(19,16,t) &= 20(5t+5)^9(5t+3)^9(5t+8) \\
P(19,17,t) &= 20(5t+5)(5t+4)^{10}((5t)^2+10(5t)+20)^4 \\
P(19,18,t) &= 20(5t)(5t+5)^{18}.
\end{aligned}$$

To obtain each formula for $P(19, r, t)$, we require a trace-minimal or bipartite-trace-minimal graph from a particular graph class. All of these are known and appear in Sections 4, 2. For each r, we will list the graph class, a trace-minimal (or bipartite-trace-minimal) graph G (B) in the graph class, the section number where the trace-minimal graph is given, and the theorem and equation from Section 1.3 used to transform the characteristic polynomial of a trace-minimal graph into a formula for $P(19, r, t)$.

r	Sec.	class	graph G or B	Theorem/Equation
0	4.1	$\mathcal{G}(20,0)$	I_{20}	1.5 (7)
1	4.1	$\mathcal{B}(10,0)$	I_{10}	1.2 (3)
2	4.1	$\mathcal{G}(10,5)$	$K_{5,5}$	1.3 (4)
3	4.1	$\mathcal{B}(20,1)$	$10K_2$	1.4 (6)
4	4.1	$\mathcal{G}(20,1)$	$10K_2$	1.5 (7)
5	4.1	$\mathcal{G}(10,1)$	$5K_2$	1.2 (3)
6	4.8.3	$\mathcal{G}(10,6)$	$C_6 \bigtriangledown I_4$	1.3 (4)
7	4.2	$\mathcal{B}(20,2)$	C_{20}	1.4 (6)
8	4.2	$\mathcal{G}(10,2)$	C_{20}	1.5 (7)
9	4.2	$\mathcal{G}(10,2)$	C_{10}	1.2 (3)
10	4.8.2	$\mathcal{G}(10,7)$	$2K_2 \bigtriangledown I_3^{(2)}$	1.3 (4)
11	4.8.2	$\mathcal{G}(20,17)$	$C_5 \bigtriangledown I_3^{(5)}$	1.4 (5)
12	4.2	$\mathcal{B}(20,8)$	$K_{10,10} - C_{20}$	1.5 (8)
13	4.3	$\mathcal{G}(10,3)$	Petersen	1.2 (3)
14	4.1	$\mathcal{G}(10,8)$	$I_2^{(5)}$	1.3 (4)
15	4.1	$\mathcal{G}(20,18)$	$I_2^{(10)}$	1.4 (5)
16	4.1	$\mathcal{B}(20,9)$	$K_{10,10} - 10K_2$	1.5 (7)
17	4.7.2	$\mathcal{G}(10,4)$	$S(10,4)$	1.2 (3)
18	4.1	$\mathcal{G}(10,9)$	K_{10}	1.3 (4)

Now it is easy to prove Theorem 5.21. Evaluate $P(19,r,t)$ from the appropriate characteristic polynomial (given in the list above) using the appropriate theorem from Section 1.3. For example, take $r = 12$. Since $r \equiv 0 \pmod 4$, we use Theorem 1.5 with $d = 3$ and $p = 5$. And since $p/2 < d$, we use the second part of Theorem 1.5 and the graph class $\mathcal{B}(4p, p+d) = \mathcal{B}(20,8)$. The bipartite graph $B = K_{10,10} - C_{20}$ is bipartite-trace-minimal (Section 4.2) and $\mathrm{ch}(B,x) = (x-8)(x+8)x^2(x^2-x-1)^2(x^2+x-1)^2(x^4-5x^2+5)^2$. We now evaluate the expression for $P(19,12,t)$ given in Equation (8) of Theorem 1.5:

$$\begin{aligned}
pt + d &= 5t + 3\\
pt + 2d &= 5t + 6\\
p(t+1) + 2d &= 5t + 11\\
P(19,12,t) &= \frac{4(5t+6)\mathrm{ch}(B,5t+3)}{(t-1)(5t+11)}\\
&= 20(5t+3)^2(5t+6)((5t)^2+5(5t)+5)^2((5t)^2+7(5t)+11)^2\\
&\quad \times((5t)^4+12(5t)^3+49(5t)^2+78(5t)+41)^2.
\end{aligned}$$

The other parts of Theorem 5.21 are proved in a similar way.

5.6 $n = 23$

Theorem 5.22

$$
\begin{array}{rcl}
P(23,0,t) & = & 24(6t)^{23} \\
P(23,1,t) & = & 24(6t)^{22}(6t+6) \\
P(23,2,t) & = & 24(6t)^{21}(6t+6)^2 \\
P(23,3,t) & = & 24(6t)^{11}(6t-4)(6t+2)^{11} \\
P(23,4,t) & = & 24(6t)^{11}(6t+2)^{12} \\
P(23,5,t) & = & 24(6t)^{10}(6t+2)^{12}(6t+6) \\
P(23,6,t) & = & 24(6t)(6t+1)^8(6t+6)^2((6t)^3+4(6t)^2+3(6t)-1)^4 \\
P(23,7,t) & = & 24(6t-2)(6t+1)^2(6t+2)^2(6t+3)^2((6t)^2+4(6t)+1)^2 \\
 & & \times((6t)^2+4(6t)+2)^2((6t)^4+8(6t)^3+20(6t)^2+16(6t)+1)^2 \\
P(23,8,t) & = & 24(6t+4)(6t+1)^2(6t+2)^2(6t+3)^2((6t)^2+4(6t)+1)^2 \\
 & & \times((6t)^2+4(6t)+2)^2((6t)^4+8(6t)^3+20(6t)^2+16(6t)+1)^2 \\
P(23,9,t) & = & 24(6t+1)^4(6t+2)^4(6t+3)^4(6t+4)^2(6t+6)((6t)^2+4(6t)+1)^4 \\
P(23,10,t) & = & 24(6t)(6t+2)^{18}(6t+6)^4 \\
P(23,11,t) & = & 24(6t+2)^{18}(6t+6)^5 \\
P(23,12,t) & = & 24(6t+1)^3(6t+3)^3(6t+4)^2(6t+5)((6t)^2+7(6t)+8)((6t)^3+10(6t)^2+29(6t)+22)^4 \\
P(23,13,t) & = & 24(6t+2)^4(6t+3)^2(6t+5)^4(6t+6)((6t)^2+6(6t)+7)^4((6t)^2+7(6t)+8)^2 \\
P(23,14,t) & = & 24(6t)(6t+3)^{16}(6t+6)^6 \\
P(23,15,t) & = & 24(6t+3)^{16}(6t+6)^7 \\
P(23,16,t) & = & 24(6t+3)^2(6t+4)^2(6t+5)^2(6t+8)((6t)^2+8(6t)+13)^2 \\
 & & \times((6t)^2+8(6t)+14)^2((6t)^4+16(6t)^3+92(6t)^2+224(6t)+193)^2 \\
P(23,17,t) & = & 24(6t+3)^{12}(6t+4)^2(6t+6)^5(6t+7)^4 \\
P(23,18,t) & = & 24(6t)(6t+4)^{12}(6t+6)^{10} \\
P(23,19,t) & = & 24(6t+4)^{12}(6t+6)^{11} \\
P(23,20,t) & = & 24(6t+4)^{11}(6t+6)^{11}(6t+10) \\
P(23,21,t) & = & 24(6t+4)^{10}(6t+6)^{11}(6t+10)^2 \\
P(23,22,t) & = & 24(6t)(6t+6)^{22}
\end{array}
$$

The corresponding trace-minimal graphs are given in Section 5.6.

In this section we list all the trace-minimal and bipartite-trace-minimal graphs that are required to obtain formulas for $P(23,r,t)$ and all $0 \le r < 23$. For each r, we will list a section number, the graph class, a trace-minimal (or bipartite-trace-minimal) graph G (B) in the graph class, and its characteristic polynomial $\mathrm{ch}(G,x)$ $(\mathrm{ch}(B,x))$. The proof that the graph is trace-minimal (bipartite-trace-minimal) in its class is given in the section listed. The proof of Theorem 5.22 follows by applying the appropriate result, which depends on the congruence class of $r \pmod 4$, from Section 1.3.

r	Sec.	class	graph G	Theorem/Equation
0	4.1	$\mathcal{G}(24,0)$	I_{24}	1.5 (7)
1	4.1	$\mathcal{G}(12,0)$	I_{12}	1.2 (3)
2	4.1	$\mathcal{G}(12,6)$	$K_{6,6}$	1.3 (4)
3	4.1	$\mathcal{B}(24,1)$	$12K_2$	1.4 (6)
4	4.1	$\mathcal{G}(24,1)$	$12K_2$	1.5 (7)
5	4.1	$\mathcal{G}(12,1)$	$6K_2$	1.2 (3)
6	4.8.4	$\mathcal{G}(12,7)$	$I_5 \bigtriangledown C_7$	1.3 (4)
7	4.2	$\mathcal{B}(24,2)$	C_{24}	1.4 (6)
8	4.2	$\mathcal{G}(24,2)$	C_{24}	1.5 (7)
9	4.2	$\mathcal{G}(12,2)$	C_{12}	1.2 (3)
10	4.8.3	$\mathcal{G}(12,8)$	$I_4^{(3)}$	1.3 (4)
11	4.8.3	$\mathcal{G}(24,20)$	$I_4^{(6)}$	1.4 (5)
12	4.5	$\mathcal{G}(24,3)$	$S(24,3)$ (McGee)	1.5 (7)
13	4.6	$\mathcal{G}(12,3)$	$S(12,3)$	1.2 (3)
14	4.8.2	$\mathcal{G}(12,9)$	$I_3^{(4)}$	1.3 (4)
15	4.8.2	$\mathcal{G}(24,21)$	$I_3^{(8)}$	1.4 (5)
16	4.2	$\mathcal{B}(24,10)$	$K_{12,12} - C_{24}$	1.5 (8)
17	4.7.2	$\mathcal{G}(12,4)$	$S(12,4)$	1.2 (3)
18	4.1	$\mathcal{G}(12,10)$	$I_2^{(6)}$	1.3 (4)
19	4.1	$\mathcal{G}(24,22)$	$I_2^{(12)}$	1.4 (5)
20	4.1	$\mathcal{B}(24,11)$	$K_{12,12} - 12K_2$	1.5 (8)
21	4.1	$\mathcal{G}(12,5)$	$K_{6,6} - 6K_2$	1.2 (3)
22	4.1	$\mathcal{G}(12,11)$	K_{12}	1.3 (4)

5.7 $n = 27$

Theorem 5.23

$$
\begin{aligned}
P(27,0,t) &= 28(7t)^{27}\\
P(27,1,t) &= 28(7t)^26(7t+7)\\
P(27,2,t) &= 28((7t)^25(7t+7)^2\\
P(27,3,t) &= 28(7t)^{13}(7t-5)(7t+2)^{13}\\
P(27,4,t) &= 28(7t)^{13}(7t+2)^{14}\\
P(27,5,t) &= 28(7t)^{12}(7t+2)^{14}(7t+7)\\
P(27,6,t) &= 28(7t+1)^{14}(7t+3)^2(7t+7)^2((7t)^2+2(7t)-1)^4\\
P(27,7,t) &= 28(7t+2)^2(7t-3)((7t)^3+5(7t)^2+6(7t)+1)^2\\
&\quad\times((7t)^3+7(7t)^2+14(7t)+7)^2((7t)^6+12(7t)^5+53(7t)^4+104(7t)^3+86(7t)^2+24(7t)+1)^2\\
P(27,8,t) &= 28(7t+2)^2(7t+4)((7t)^3+5(7t)^2+6(7t)+1)^2\\
&\quad\times((7t)^3+7(7t)^2+14(7t)+7)^2((7t)^6+12(7t)^5+53(7t)^4+104(7t)^3+86(7t)^2+24(7t)+1)^2\\
P(27,9,t) &= 28(7t+4)^2(7t+7)((7t)^3+5(7t)^2+6(7t)+1)^4((7t)^3+7(7t)^2+14(7t)+7)^4\\
P(27,10,t) &= 28(7t)(7t+1)^2(7t+2)^{10}(7t+3)^4(7t+4)^4(7t+7)^2((7t)^2+3(7t)-2)^2\\
P(27,11,t) &= 28(7t-1)(7t+2)(7t+4)((7t)^3+9(7t)^2+23(7t)+14)^4((7t)^3+9(7t)^2+23(7t)+16)^4\\
P(27,12,t) &= 28(7t+1)^3(7t+3)^6(7t+5)^5((7t)^2+6(7t)+7)\\
&\quad\times((7t)^3+10(7t)^2+27(7t)+16)((7t)^3+10(7t)^2+29(7t)+22)\\
P(27,13,t) &= 28(7t+7)(7t+6)^2((7t)^2+6(7t)+7)^{12}\\
P(27,14,t) &= 28(7t)(7t+2)^4(7t+3)^{12}(7t+4)^4(7t+5)^2(7t+7)^4\\
P(27,15,t) &= 28(7t+3)^{21}(7t+7)^6\\
P(27,16,t) &= 28(7t+3)(7t+5)(7t+8)((7t)^3+12(7t)^2+44(7t)+49)^4((7t)^3+12(7t)^2+44(7t)+47)^4\\
P(27,17,t) &= 28(7t+3)^6(7t+4)^4(7t+5)^2(7t+7)^3((7t)^2+9(7t)+16)^6\\
P(27,18,t) &= 28(7t)(7t+4)^{12}(7t+7)^6((7t)^2+9(7t)+19)^4\\
P(27,19,t) &= 28(7t+3)(7t+4)^{16}(7t+5)^2(7t+7)^8\\
P(27,20,t) &= 28(7t+5)^2(7t+10)((7t)^3+14(7t)^2+63(7t)+91)^2\\
&\quad\times((7t)^3+16(7t)^2+83(7t)+139)^2\\
&\quad\times((7t)^6+30(7t)^5+368(7t)^4+2360(7t)^3+8339(7t)^2+15390(7t)+11593)^2\\
P(27,21,t) &= 28(7t+3)^2(7t+4)^4(7t+5)^8(7t+6)^2(7t+7)^3((7t)^2+13(7t)+34)^2((7t)^2+13(7t)+38)^2\\
P(27,22,t) &= 28(7t)(7t+5)^{14}(7t+7)^{12}\\
P(27,23,t) &= 28(7t+5)^{14}(7t+7)^{13}\\
P(27,24,t) &= 28(7t+5)^{13}(7t+7)^{13}(7t+12)\\
P(27,25,t) &= 28(7t+5)^{12}(7t+7)^{13}(7t+12)^2\\
P(27,26,t) &= 28(7t)(7t+7)^{26}
\end{aligned}
$$

To obtain each formula for $P(27,r,t)$, we require a trace-minimal or bipartite-trace-minimal graph from a particular graph class. All of these are known and appear in Section 4. For each r, we will list the graph class, a trace-minimal (or bipartite-trace-minimal) graph G (B) in the graph class, the section number where the trace-minimal graph is given, and the theorem and equation from Section 1.3 used to transform the characteristic polynomial of a trace-minimal graph into a formula for $P(27,r,t)$.

r	Sec.	class	graph G or B	Theorem/Equation
0	4.1	$\mathcal{G}(28,0)$	I_{28}	1.5 (7)
1	4.1	$\mathcal{G}(14,0)$	I_{14}	1.2 (3)
2	4.1	$\mathcal{G}(14,7)$	$K_{7,7}$	1.3 (4)
3	4.1	$\mathcal{B}(28,1)$	$14K_2$	1.4 (6)
4	4.1	$\mathcal{G}(28,1)$	$14K_2$	1.5 (7)
5	4.1	$\mathcal{G}(14,1)$	$7K_2$	1.2 (3)
6	4.8.5	$\mathcal{G}(14,8)$	$I_6 \bigtriangledown C_8$	1.3 (4)
7	4.2	$\mathcal{B}(28,2)$	C_{28}	1.4 (6)
8	4.2	$\mathcal{G}(28,2)$	C_{28}	1.5 (7)
9	4.2	$\mathcal{G}(14,2)$	C_{14}	1.2 (3)
10	4.8.4	$\mathcal{G}(14,9)$	$S(9,4) \bigtriangledown I_5$	1.3 (4)
11	4.6	$\mathcal{B}(28,3)$	$SB(28,3)$	1.4 (6)
12	4.6	$\mathcal{G}(28,3)$	$S(28,3)$	1.5 (7)
13	4.4.1	$\mathcal{G}(14,3)$	$PP(2) =$Heawood	1.2 (3)
14	4.8.3	$\mathcal{G}(14,10)$	$C_6 \bigtriangledown I_4^{(2)}$	1.3 (4)
15	4.8.3	$\mathcal{G}(28,24)$	$I_4^{(7)}$	1.4 (5)
16	4.6	$\mathcal{B}(28,11)$	$SB(28,11)$	1.5 (8)
17	4.6	$\mathcal{G}(14,4)$	$S(14,4)$	1.2 (3)
18	4.8.2	$\mathcal{G}(14,11)$	$C_5 \bigtriangledown I_3^{(3)}$	1.3 (4)
19	4.8.2	$\mathcal{G}(28,25)$	$2K_2 \bigtriangledown I_3^{(8)}$	1.4 (5)
20	4.2	$\mathcal{B}(28,12)$	$K_{14,14} - C_{28}$	1.5 (8)
21	4.6	$\mathcal{G}(14,5)$	$S(14,5)$	1.2 (3)
22	4.1	$\mathcal{G}(14,12)$	$I_2^{(7)}$	1.3 (4)
23	4.1	$\mathcal{G}(28,26)$	$I_2^{(14)}$	1.4 (5)
24	4.1	$\mathcal{B}(28,13)$	$K_{14,14} - 14K_2$	1.5 (8)
25	4.1	$\mathcal{G}(14,6)$	$K_{7,7} - 7K_2$	1.2 (3)
26	4.1	$\mathcal{G}(14,13)$	K_{14}	1.3 (4)

6 Proofs that sporadic graphs are trace-minimal

In this section we outline the proofs that each of the sporadic graphs given in Section 4.6 is the unique trace-minimal graph in its graph class.

We begin by giving complete proofs for the graphs $S(8,3), S(9,4), S(10,4)$ since these graphs are used to prove Theorems 5.16, 5.17, 5.18, 5.19.

Lemma 6.1 *The graph $S(8,3)$ is the only trace-minimal graph in $\mathcal{G}(8,3)$.*

Proof: It is easy to see that $S(8,3)$ has four 4-cycles but no 3-cycles. Thus the girth of $S(8,3)$ is four and by Theorem 2.1, it is enough to show that every other graph $G \in \mathcal{G}(8,3)$ either has a 3-cycle or has more than four 4-cycles.

Assume G has no 3-cycles. If G has no odd cycles then G is bipartite, and thus $G = K_{4,4} - 4K_2$, since $G = K_{4,4} - 4K_2$ is the only bipartite graph in $\mathcal{G}(8,3)$. But $G = K_{4,4} - 4K_2$ has six 4-cycles, so it is not trace-minimal.

Now assume G has an odd cycle. Then there must be a 5-cycle, otherwise the smallest cycle would have length 7 and all the remaining edges would be incident with the last vertex. Assume vertices 1,2,3,4,5 form a 5-cycle in G. There are twelve edges in G; five connect vertices 1,2,3,4,5 to the remaining vertices, 6,7,8; and say vertex 7 is connected to 6 and 8 and to 3. Thus there are four more edges to add to the graph shown in Figure 8.

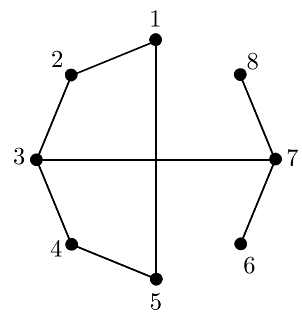

Figure 8: Incomplete trace-minimal graph G in $\mathcal{G}(8,3)$

Vertex 1 must be adjacent to either 6 or 8. By symmetry, we may assume that 8 and 1 are adjacent. Now 8 must also be adjacent to either 2,4,5, or 6. But since there are no triangles in G, 8 must be adjacent to 4, and 6 must be adjacent to 5 and 2. That is, $G = S(8,3)$. ∎

Lemma 6.2 *The graph $S(9,4)$ is the only trace-minimal graph in $\mathcal{G}(9,4)$.*

Proof: It is easy to see that $\text{cyc}(S(9,4),3) = 2$. Since the girth of $S(9,4)$ is three, then by Theorem 2.1 it is enough to show that any $G \in \mathcal{G}(9,4)$, $G \neq S(9,4)$ satisfies $\text{cyc}(G,3) > 2$.

Let $G \in \mathcal{G}(9,4)$, by Lemma 4.22, we know $\text{cyc}(G,3) \geq 1$. Suppose $\text{cyc}(G,3) = 1$, let $G_1 = \{1,2,3\}$ be the only 3 -cycle. There are six edges connecting G_1 to the remaining vertices $G_2 = \{4,5,\ldots,9\}$. Moreover, since G_1 is the only 3-cycle in G, each vertex in G_2 is adjacent to at most one vertex in G_1; so we may assume that vertices $1,2$ and, 3 are adjacent to $\{4,5\}, \{6,7\}$, and $\{8,9\}$, respectively. The induced subgraph obtained from G_2 is regular of degree 3 and has no triangles. Hence G_2 is isomorphic to $K_{3,3}$ and thus G_2^{comp} is isomorphic to $2K_3$. But since G has no 3-cycles, $(4,5)$, $(6,7)$, and $(8,9)$ are disjoint edges of $G_2^{\text{comp}} = 2K_3$, a contradiction.

Now we prove that if $\text{cyc}(G,3) = 2$ then G is isomorphic to $S(9,4)$. First, suppose the two 3-cycles share an edge, let G_1 be the subgraph induced by these four vertices and G_2 the subgraph induced by the remaining five vertices. Then G_1 has five edges, there are six edges connecting G_1 to G_2, and there are seven edges in G_2. This is impossible since G_2 is triangle free and has only five vertices. Similarly, if the two 3-cycles are disjoint then by letting G_1 be the graph induced by them and G_2 the subgraph induced by the remaining three vertices, we deduce that each vertex in G_2 is connected to at most one vertex in each 3-cycle, therefore its two remaining edges must go to the other two vertices in G_2, which would force an additional triangle in G_2.

Finally assume that vertices $1, 3, 4$ and $1, 2, 5$ are the two triangles in G. There are eight edges connecting these vertices to the remaining four vertices, therefore there are four edges in $\{6, 7, 8, 9\}$ and since there are no more 3-cycles they must form a 4-cycle as shown (without loss of generality) in Figure 9. We may also assume that vertex 6 is adjacent to 3 and 5.

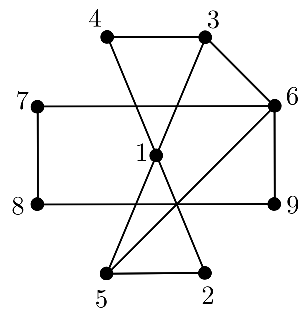

Figure 9: Incomplete trace-minimal graph G in $\mathcal{G}(9, 4)$

Now there is only one way to complete the graph avoiding 3-cycles. Namely, 7 must be adjacent to 2 and 4, 8 must be adjacent to 3 and 5, and 9 must be adjacent to 2 and 4. Thus $G = S(9, 4)$. ∎

Lemma 6.3 *The graph $S(10, 4)$ is the only trace-minimal graph in $\mathcal{G}(10, 4)$.*

Proof: It is easy to see that $\text{cyc}(S(10, 4), 3) = 0$ and $\text{cyc}(S(10, 4), 4) = 25$. Since the girth of $S(10, 4)$ is four, then (by Theorem 2.1) it is enough to show that any $G \in \mathcal{G}(10, 4)$ with $G \neq S(10, 4)$ satisfies $\text{cyc}(G, 3) > 0$, or $\text{cyc}(G, 3) = 0$ and $\text{cyc}(G, 4) > 25$.

Assume $\text{cyc}(G, 3) = 0$. If G has no odd cycles then G is bipartite, and thus $G = K_{5,5} - 5K_2$, and then $\text{cyc}(G, 4) = 30$ and G is not trace-minimal.

Next assume that G has a cycle of odd length. Let C be the cycle in G of shortest odd length c. The subgraph C has only c edges, for if there were more G would have a cycle of odd length shorter than c. Since the degree of regularity of G is four, it follows that there are $2c$ edges from the vertices of C to the other $10 - c$ vertices. Thus there are $3c$ edges with at least one vertex in C. Since there are 20 edges in G, we have $3c \leq 20$ and so $c = 5$.

Finally, assume vertices $1, 2, 3, 4, 5$ form a 5-cycle in G. There are 10 edges connecting these vertices to the remaining vertices $6, 7, 8, 9, 10$, therefore the subgraph with vertices $6, 7, 8, 9, 10$ is regular of degree 2. Hence it is a 5-cycle. Since there are no 3-cycles in G, vertex 1 must be adjacent to two non-adjacent vertices among $6, 7, 8, 9, 10$, say vertex 1 is adjacent to 7 and 10, and neither vertex 2 nor 5 is adjacent to either 7 or 10. So the remaining edges from vertices 2 and 5 (four edges in all) must be connected to vertices 6,8,9. But if either vertex 2 or vertex 5 is adjacent to both vertices 8 and 9, then there will be a 3-cycle. Thus both 2 and 5 are adjacent to vertex 6 as shown in Figure 10.

Vertex 3 must be adjacent to two non-adjacent vertices among $7, 8, 9, 10$. Since the map that switches vertex 7 with 10, 8 with 9, and fixes the remaining vertices is an automorphism of the graph in Figure 10, we may assume that vertex 3 is adjacent to 7 and 9. Now there is only one way to add the remaining four edges without creating a 3-cycle: vertex 4 must be adjacent to 8 and 10, 2 is adjacent to 8, and 5 is adjacent to 9. Thus $G = S(10, 4)$. ∎

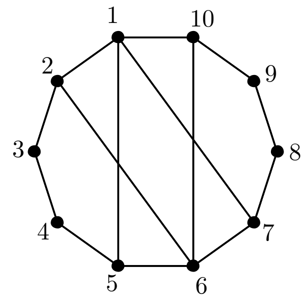

Figure 10: Incomplete trace-minimal graph G in $\mathcal{G}(10,4)$

The proofs that the next group of sporadic graphs are trace-minimal rely on a computer search. The case $S(12,4)$ is typical so we use it to explain the technique.

Lemma 6.4 *The following graphs are the only trace-minimal graphs in their graph class:*

$S(11,4)$, $S(12,4)$, $S(13,4)$, $S(13,6)$, $S(14,4)$, $S(14,5)$, $S(16,6)$, $S(20,8)$, $SB(28,3)$, $SB(28,11)$.

Proof: The proof is by computer search. It is, however, difficult to find all graphs in a given graph class without first limiting the possibilities. We do this by arguing that a trace-minimal graph must have maximum girth g and the fewest number of cycles of length g in the graph class. We demonstrate the method for the graph $S(12,4)$ in $\mathcal{G}(12,4)$. Proofs of the other cases are similar.

Let $G \in \mathcal{G}(12,4)$ be a trace-minimal graph. First note that $S(12,4)$ has girth 4 and $\mathrm{cyc}(S(12,4),4) = 15$. If the girth of G exceeds 4, then there are 12 distinct vertices whose distance from a given vertex is two. Hence G would have at least 13 vertices. It follows that G has girth at most 4. So G must have girth 4 and $\mathrm{cyc}(G,4) \leq 15$. Let $d_4(i)$ denote the number of 4-cycles in G that contain vertex i. Since

$$15 \geq \mathrm{cyc}(S(12,4),4) = \frac{1}{4}\sum_{i=1}^{12} d_4(i),$$

there must be a vertex, say vertex 1, such that $d_4(1) \leq 60/12 = 5$. Let $A = \{2,3,4,5\}$ be the vertices adjacent to vertex 1 and $B = \{6,7,8,9,10,11,12\}$ be the remaining vertices. Consider the bipartite graph H with vertices $A \cup B$ whose edges are all the edges in G from A to B. Then all vertices in H have degree at most 4 and all vertices in A have degree 3. Note that a vertex in B of degree k in H generates $\binom{k}{2}$ 4-cycles in G containing vertex 1. Since $d_4(1) \leq 5$, the only possible degree sequence corresponding the vertices of H in B is $(2,2,2,2,2,1,1)$. A computer program generated all graphs satisfying all these conditions and found that $S(12.4)$ is the only one. ∎

Finally we show that $K_{9,9} - C_{18}$ is the only trace minimal graph in $\mathcal{G}(18,7)$. This graph is not sporadic, but it requires a special argument since the techniques used in the proof of the other cases of Theorem 4.15 are insufficient.

Lemma 6.5 *The graph $K_{9,9} - C_{18}$ is the only trace minimal graph in $\mathcal{G}(18,7)$.*

Proof: Let G be a trace-minimal graph in $\mathcal{G}(18,7)$. Since $K_{9,9} - C_{18}$ has no 3-cycles, G has no 3-cycles.

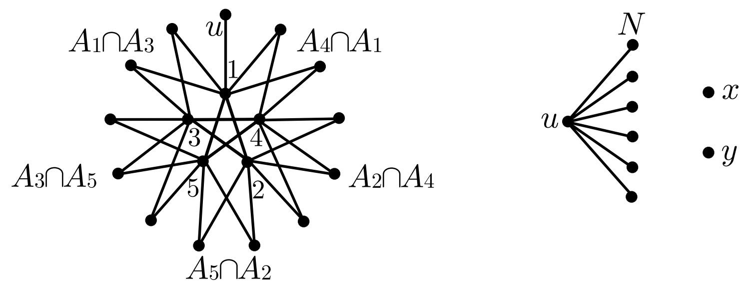

Figure 11: Incomplete trace-minimal graph G in $\mathcal{G}(18,7)$

Also, since $K_{9,9} - C_{18}$ is the unique bipartite-trace-minimal graph in $\mathcal{G}(18,7)$ (Lemma 4.2), it is enough to prove that G is bipartite.

By Lemma 4.13, if G is not bipartite and has no 3-cycles then there is a 5-cycle in G. We use the notation in such proof: we label the vertices of the 5-cycle 1 through 5 and define A_i to be the set of neighbors of i excluding $i \pm 1$. Then $|A_i| = 5$ and $A_i \cap A_{i+1} = \emptyset$. Thus

$$\begin{aligned}13 \geq\ & |A_1| + |A_2| + |A_3| + |A_4| + |A_5| \\ & - |A_1 \cap A_3| - |A_3 \cap A_5| - |A_5 \cap A_2| - |A_2 \cap A_4| - |A_4 \cap A_1|\end{aligned}$$

and then

$$|A_1 \cap A_3| + |A_3 \cap A_5| + |A_5 \cap A_2| + |A_2 \cap A_4| + |A_4 \cap A_1| \geq 12. \tag{20}$$

Also, from the proof of Lemma 4.13 we know that $A_1 \cap A_3$, $A_3 \cap A_5$, $A_5 \cap A_2$, $A_2 \cap A_4$ and $A_4 \cap A_1$ are mutually disjoint and each contains at least two vertices. Observe now that each contains at most three vertices. Indeed, $A_3 \cap A_5$ and $A_5 \cap A_2$ are disjoint and contained in A_5. So either both have two elements or one has two elements and the other three. Hence, by Inequality (20) we can assume without loss of generality that

$$\begin{aligned}&|A_1 \cap A_3| = |A_5 \cap A_2| = |A_4 \cap A_1| = 2, \text{ and} \\ &|A_3 \cap A_5| = |A_2 \cap A_4| = 3.\end{aligned}$$

We get the subgraph of G shown in Figure 11 (left) (u is the only vertex belonging to exactly one A_i).

Since there are no 3-cycles in G, there are no edges between $A_1 \cap A_3$ and $(A_3 \cap A_5) \cup (A_4 \cap A_1)$. Since the vertices of $A_1 \cap A_3$ are still missing five neighbors, then all vertices in $A_1 \cap A_3$ are adjacent to all vertices in $A_2 = (A_2 \cap A_4) \cup (A_5 \cap A_2)$. Similarly, all vertices in $A_4 \cap A_1$ are adjacent to all vertices in $A_5 = (A_3 \cap A_5) \cup (A_5 \cap A_2)$.

All remaining 13 edges must join vertices in $U = \{u\} \cup (A_3 \cap A_5) \cup (A_2 \cap A_4) \cup (A_5 \cap A_2)$. Let H be the subgraph of G induced by U. H satisfies that u has degree 6 and all other vertices have degree at most 3. Let N be the set of vertices adjacent to u in H. There are only two vertices in $U - (N \cup \{u\})$, say x and y. See Figure 11 (right). Since H has no 3-cycles then all remaining edges must have at least one vertex in $\{x, y\}$. So $13 \leq 6 + \deg_H x + \deg_H y \leq 12$. Then G is bipartite. ∎

Finally we prove deal with the trace-minimal graphs in $\mathcal{G}(14,8)$ and $\mathcal{G}(16,10)$ that are needed in Section 4.9.4 for the proof of Lemma 4.19.

Lemma 6.6 *The graph $I_6 \bigtriangledown C_8$ is trace-minimal in $\mathcal{G}(14,8)$. The graph $I_6 \bigtriangledown S(10,4)$ is trace-minimal in $\mathcal{G}(16,10)$.*

First, we establish the following lemma.

Lemma 6.7 *If $G \in \mathcal{G}(v,5)$ and $\triangle(G,i) \leq 8$ for every vertex i in G, then G has at most $3\lfloor v/6 \rfloor$ vertices j with $\triangle(G,j) = 8$.*

Proof: Suppose $\{1,2,\ldots,v\}$ are the vertices of G. Assume $\triangle(G,1) = 8$. Let $A = \{2,3,4,5,6\}$ be the vertices adjacent to 1. Since $\triangle(G,1) = 8$ there are only two pairs in A that are not edges in G. We have two cases.

First suppose that $(2,3)$ and $(2,4)$ are the only edges not present in G. Let $B = \{1,5,6,7,8\}$ be the vertices adjacent to 2. We have that $\triangle(G,5) = \triangle(G,6) = 8$. Note that $\triangle(G,2)$ equals the number of edges connecting vertices of B, and since 7 and 8 are not connected to $1,5$ or 6, then $\triangle(G,2) \leq 4$. In addition we have that $\triangle(G,7) \leq 7$, since at least three of the vertices adjacent to 7 are not adjacent to 2. Similarly, if $C = \{1,4,5,6,x\}$ are the vertices adjacent to 3, then $\triangle(G,3) \leq 7$ since x is not connected to $1,5$ or 6. Again we have that $\triangle(G,x) \leq 6$ since at least three of the vertices adjacent to x are not adjacent to 3. By symmetry $\triangle(G,4) \leq 7$.

For the second case suppose $(2,3)$ and $(4,5)$ are the only edges not in G. Let $B = \{1,4,5,6,7\}$ be the vertices adjacent to 2. Since 7 is not connected to 1 or 6 then $\triangle(G,2) \leq 7$. Observe that we also have that $\triangle(G,7) \leq 6$ since at least three of the vertices adjacent to 7 are not adjacent to 2. By symmetry we also have $\triangle(G,i) \leq 7$ for $i = 3,4,5$.

Finally, note that in both cases we proved that either two vertices i_1 and i_2 with $\triangle(G,i_1) = \triangle(G,i_2) = 8$ are adjacent, or else they have no common adjacent vertices. Thus there are at most $\lfloor v/6 \rfloor$ pairwise-non-adjacent vertices i with $\triangle(G,i) = 8$. Moreover, each of these is connected to at most two other vertices j with $\triangle(G,j) = 8$ (this only happens in the first case). Thus there are at most $3\lfloor v/6 \rfloor$ vertices j with $\triangle(G,j) = 8$. ∎

We can know prove Lemma 6.6. Let $H \in \mathcal{G}(14,8)$ be trace-minimal. By Equation (14), H^{comp} has the largest number of triangles in $\mathcal{G}(14,5)$. Then by Theorem 4.20, either $\triangle(H^{\text{comp}},i) \leq 8$ for all vertices i in H^{comp} or $K_6 \subseteq H^{\text{comp}}$. In the first case, by Lemma 6.7, there are at most 6 vertices j with $\triangle(G,j) = 8$. Thus

$$\operatorname{cyc}(H^{\text{comp}},3) = \frac{1}{3}\sum_{i=1}^{14} \triangle(H^{\text{comp}},i) \leq \frac{1}{3}(6 \cdot 8 + 8 \cdot 7) < 35.$$

But $\operatorname{cyc}((I_6 \bigtriangledown C_8)^{\text{comp}},3) = 36$, which contradicts that H is trace-minimal. If, on the other hand, $K_6 \subseteq H^{\text{comp}}$ then $H = I_6 \bigtriangledown H_1$ where $H_1 \in \mathcal{G}(8,2)$ is trace-minimal. By Lemma 4.2, $H_1 = C_8$ and then $H = I_6 \bigtriangledown C_8$.

Similarly, if $H \in \mathcal{G}(16,10)$ is trace-minimal then by Equation (14), H^{comp} has the largest number of triangles in $\mathcal{G}(16,5)$. Again by Theorem 4.20, either $\triangle(H^{\text{comp}},i) \leq 8$ for all vertices i in H^{comp} or $K_6 \subseteq H^{\text{comp}}$. In the first case, by Lemma , there are at most 6 vertices j with $\triangle(G,j) = 8$. Thus

$$\operatorname{cyc}(H^{\text{comp}},3) = \frac{1}{3}\sum_{i=1}^{16} \triangle(H^{\text{comp}},i) \leq \frac{1}{3}(6 \cdot 8 + 10 \cdot 7) < 40.$$

But $\mathrm{cyc}((I_6 \bigtriangledown S(10,4))^{\mathrm{comp}}, 3) = 40$, which contradicts that H is trace-minimal. If, on the other hand, $K_6 \subseteq H^{\mathrm{comp}}$ then $H = I_6 \bigtriangledown H_1$ where $H_1 \in \mathcal{G}(10,4)$ is trace-minimal. By Lemma 6.3, $H_1 = S(10,4)$ and then $H = I_6 \bigtriangledown S(10,4)$.

7 Proofs of Theorems 2.1 and 2.3

The proofs of Theorems 2.1 and 2.3 depend on an application of the Coefficient Theorem [Sa1], [CDS, Theorem 1.3] to regular graphs [Sa2], [CDS, Theorem 3.26]. We summarize these results now. Let H be a regular graph in $\mathcal{G}(v,\delta)$ with girth h and let $x^v + a_1x^{v-1} + \cdots + a_v$ be the characteristic polynomial of $A(H)$. Then the coefficient a_i for $i \leq 2h-1$ depends only on the numbers of cycles in H, $\mathrm{cyc}(H,j)$, with $j \leq i$ and not on the particular structure of the graph H. Indeed $a_1 = 0$ and $a_2 = v\delta$. For $i \geq 3$, there exist integers $u(q,i)$ for $q = 0$, $3 \leq q \leq i$, and $i+q$ even, such that for every $H \in \mathcal{G}(v,\delta)$ with girth h and $i \leq 2h-1$,

$$a_i \;=\; (-1)^{\frac{i}{2}}u(0,i) - 2\mathrm{cyc}(H,i) - 2\sum(-1)^{\frac{i+q}{2}}\mathrm{cyc}(H,q)u(q,i), \tag{21}$$

where $u(0,i) = 0$ if i is odd, and the sum is taken over integers q satisfying $h \leq q < i$ and $i+q$ even.

We also require Newton's Identities [BP] in matrix form. Let X be a $v \times v$ matrix with characteristic polynomial $\det(xI - X) = x^v + c_1x^{v-1} + c_2x^{v-2} + \cdots c_n$. Then for all $q \leq v$,

$$0 = \mathrm{tr}X^q + c_1\mathrm{tr}X^{q-1} + c_2\mathrm{tr}X^{q-2} + \cdots + qc_q.$$

Proof: Theorem 2.1 Suppose that the graph $G \in \mathcal{G}(v,\delta)$ satisfies the hypotheses of the theorem. Let H be a graph in $\mathcal{G}(v,\delta)$ with girth h, and let

$$\begin{aligned}\mathrm{ch}(G,x) &= x^v + b_1x^{v-1} + \cdots + b_v,\\ \mathrm{ch}(H,x) &= x^v + a_1x^{v-1} + \cdots + a_v.\end{aligned}$$

By the hypothesis, $\mathrm{cyc}(G,q) = \mathrm{cyc}(H,q)$ for all $q < k$ and $\mathrm{cyc}(G,k) < \mathrm{cyc}(H,k)$. Since G has the maximum girth in $\mathcal{G}(v,\delta)$, $h \leq g$. There are two cases.

Case I $h < g$: In this case $\mathrm{cyc}(H,q) = \mathrm{cyc}(G,q) = 0$ for all $q < h$, $\mathrm{cyc}(G,h) = 0$, and $\mathrm{cyc}(H,h) > 0$. (Thus $k = h$.) Applying Equation (21) to H, G and $i < h$, we get

$$a_i = b_i = (-1)^{\frac{i}{2}}u(0,i).$$

Thus from Newton's Identities we have $\mathrm{tr}A(H)^i = \mathrm{tr}A(G)^i$ for $i < h$.

For $i = h$, Equation (21) gives

$$\begin{aligned}a_h &= (-1)^{\frac{h}{2}}u(0,h) - 2\mathrm{cyc}(H,h),\\ b_h &= (-1)^{\frac{h}{2}}u(0,h).\end{aligned}$$

Since $\mathrm{cyc}(H,h) > 0$, $a_h < b_h$. By Newton's Identities $\mathrm{tr}A(G)^h < \mathrm{tr}A(H)^h$. Thus G is trace-dominated by H.

Case II $h = g$: In this case $k \geq g$. Let $i < k$. Then $\mathrm{cyc}(H,q) = \mathrm{cyc}(G,q)$ for all $q < i$ so the expressions in Equation (21) for a_i and b_i are identical. Thus $a_i = b_i$ and it follows from Newton's Identities that $\mathrm{tr}A(H)^i = \mathrm{tr}A(G)^i$ for $i < k$.

The only place where the expressions in Equation (21) for a_k and b_k differ is $-2\mathrm{cyc}(H,k)$ and $-2\mathrm{cyc}(G,k)$. Since $\mathrm{cyc}(G,k) < \mathrm{cyc}(H,k)$, we have $a_k < b_k$. Thus from Newton's Identities we have $\mathrm{tr}A(G)^k < \mathrm{tr}A(H)^k$. So G is trace-dominated by H. ∎

To prove Theorem 2.3 we need the following Lemma:

Lemma 7.1 *Let G, H be graphs in $\mathcal{G}(v,\delta)$. Suppose G is connected and has $k+1$ eigenvalues. If $\mathrm{tr}A(G)^i = \mathrm{tr}A(H)^i$ for $i = 2,\ldots,2k-1$ then either $\mathrm{spec}(A(G)) = \mathrm{spec}(A(H))$ or $\mathrm{tr}A(G)^{2k} < \mathrm{tr}A(H)^{2k}$, which implies that G is trace-dominated by H.*

Proof: Theorem 2.3 Let $G \in \mathcal{G}(v,\delta)$ satisfy the hypotheses of the theorem. Let $H \in \mathcal{G}(v,\delta)$ have girth h. We show that G is trace-dominated by H.

If $h < g$, then the same argument as in Case I of the proof of Theorem 2.1 shows that G is trace-dominated by H.

Now suppose that $h \geq g$. Then for $i < g$, $\mathrm{cyc}(H,i) = \mathrm{cyc}(G,i) = 0$ and hence from Equation (21) and Newton's Identities we have $\mathrm{tr}A(H)^i = \mathrm{tr}A(G)^i$. If g is even, then $g = 2k$ and if g is odd then $g = 2k+1$. Either way $i \leq 2k$ for all $i < g$. Thus by Lemma 7.1 G is trace-dominated by H. ∎

All that remains is the proof of Lemma 7.1. We need the following lemma:

Lemma 7.2 *Let λ and μ be multi-sets of real numbers with cardinality N. Suppose λ has k distinct values $\lambda_1,\ldots,\lambda_k$ with multiplicities $n_i > 0$ and μ has h distinct values $\mu_1,\ldots,\mu_h$ with multiplicities $m_i > 0$. If $\sum_{i=1}^k n_i\lambda_i^j = \sum_{i=1}^h m_i\mu_i^j$ for $j = 1,\ldots,2k-1$ then, $\sum_{i=1}^k n_i\lambda_i^{2k} \leq \sum_{i=1}^h m_i\mu_i^{2k}$, with equality if and only if $\lambda = \mu$.*

Proof:

Define a polynomial $q(y) = (y-\lambda_1)\cdots(y-\lambda_k)$. Clearly,

$$q(y)^2 = y^{2k} + h_1 y^{2k-1} + \cdots + h_{2k},$$

where h_i is an integral polynomial in $\lambda_1,\ldots,\lambda_k$. Let $x = (x_1,\ldots,x_N)$, where x_i are independent indeterminates and define a polynomial by

$$f(x_1,\ldots,x_N) = \sum_{t=1}^N q(x_t)^2.$$

Then

$$f(x) = S_{2k}(x) + h_1 S_{2k-1}(x) + \cdots + N h_{2k},$$

where $S_j(x) = \sum x_t^j$ is the jth power sum of x. Clearly $f(\lambda) = 0$ and since $S_j(\lambda) = S_j(\mu)$ for $j = 1,\ldots,2k-1$, we have

$$f(\mu) = f(\mu) - f(\lambda) = S_{2k}(\mu) - S_{2k}(\lambda).$$

But $f(\mu) \geq 0$ since it is a sum of squares of real numbers. It follows that $S_{2k}(\lambda) \leq S_{2k}(\mu)$.

Now suppose $S_{2k}(\lambda) = S_{2k}(\mu)$. Then $f(\mu) = 0$ and so the distinct values of μ are among the distinct values $\lambda_1,\ldots,\lambda_k$ of λ. In particular $h \leq k$ and by interchanging the roles of λ and μ, we get $h = k$. So we may assume that $\mu_i = \lambda_i$ for $i = 1,\ldots,k$.

We have $\sum_{i=1}^{k}(m_i - n_i)\lambda_i^j = 0$ for $j = 1, \ldots, k$. But the van der Monde matrix based on λ_i is invertible and so $m_i = n_i$, for $i = 1, \ldots, k$. that is, $\lambda = \mu$. ∎

We now apply Lemma 7.2 to the reduced spectrum of the adjacency matrix of a graph to finish the proof of Lemma 7.1.

Let $G \in \mathcal{G}(v, \delta)$ be connected with adjacency matrix $A(G)$ such that $\mathrm{spec}(A(G))$ has $k + 1$ distinct values. Since G is δ-regular and connected, δ is a simple eigenvalue of $A(G)$. Thus the reduced spectrum $\mathrm{spec}'(A(G)) = \lambda$ of $A(G)$ has k distinct eigenvalues λ_i with multiplicity n_i for $i = 1, \ldots, k$, where $\sum_i n_i = v - 1$. Let H be a δ-regular graph with adjacency matrix $A(H)$ with reduced spectrum μ having h distinct eigenvalues $\mu_1, \ldots, \mu_h$ with multiplicities $m_1, \ldots, m_h$. Denote the power-sums of λ and μ by

$$\begin{array}{rcccl} S_j(\lambda) & = & \mathrm{tr}A(G)^j - d^j & = & \sum_{i=1}^{k} n_i\, \lambda_i^j \\ S_j(\mu) & = & \mathrm{tr}A(H)^j - d^j & = & \sum_{i=1}^{h} m_i\, \mu_i^j. \end{array}$$

By the hypothesis of the lemma, $S_j(\lambda) = S_j(\mu)$ for $1 \leq j \leq 2k - 1$. It follows from Lemma 7.2 that $S_{2k}(\lambda) \leq S_{2k}(\mu)$. If $S_{2k}(\lambda) < S_{2k}(\mu)$, we are finished. Otherwise $S_{2k}(\lambda) = S_{2k}(\mu)$ and then it follows from Lemma 7.2 that $\lambda = \mu$, that is $A(G)$ and $A(H)$ have the same spectrum. The proof of Lemma 7.1 is complete.

References

[AFNW] B. Ábrego, S. Fernández-Merchant, M. G. Neubauer, and W. Watkins, D-optimal weighing designs for $n \equiv -1 \pmod 4$ objects and a large number of weighings, *Linear Algebra Appl.*, 374 (2003) 175-218.

[Bri] G. Brinkmann, Fast generation of cubic graphs, *J. Graph Theory*, 23 (1996) 139-149.

[BP] W.S. Burnside, A.W. Panton, *The Theory of Equations with an Introduction to the Theory of Binary Algebraic Forms*, Dover, New York, 1960.

[Co] J.H.E. Cohn, Determinants with elements ± 1, *J. London Math. Soc.*, 42 (1967) 436-442.

[CDS] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Application*, Academic Press, New York, 1980.

[CRC] *The CRC Handbook of Combinatorial Design*, Eds. C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, 1996.

[FG] H.-J. Finck, G. Grohmann, Vollständiges Produkt, chromatische Zahl und charakteristisches Polynom regulärer Graphen, *I. Wiss. Z. TH Ilmenau*, 11 (1965) 1-3.

[God] C.D. Godsil, Problems in algebraic combinatorics, *Electron. J. Combin.* F1 (1995).

[Gor] Gordon's Data. www.cs.uwa.edu.au/~gordon/data.html

[Ho] H. Hotelling, Some improvements in weighing and other experimental techniques, *Ann. Math. Statist.*, 15 (1944) 297-306.

[HKL] M. Hudelson, V. Klee and D. Larman, Largest j-simplices in d-cubes: Some relatives of the Hadamard determinant problem, *Linear Algebra Appl.*, 241 (1996) 519-598.

[Mo] A.M. Mood, On Hotelling's weighing problem, *Ann. Math. Statist.*, 17 (1946), 432-446.

[NR] M. Neubauer and A.J. Radcliffe, The maximum determinant of (± 1)-matrices, *Linear Algebra Appl.*, 257 (1997) 289-306.

[NWZ1] M. Neubauer, W. Watkins and J. Zeitlin, Notes on D-optimal designs, *Linear Algebra Appl.*, 280 (1998) 109-127.

[NWZ2] M. Neubauer, W. Watkins and J. Zeitlin, Maximal D-optimal weighing designs for 4 and 5 objects, *Electron. J. Linear Algebra*, 4 (1998) 48-72, http://math.technion.ac.il/iic/ela.

[NWZ3] M. Neubauer, W. Watkins and J. Zeitlin, D-optimal weighing designs for six objects, *Metrika*, 52 (2000) 185-211.

[NW] M. Neubauer and W. Watkins, D-optimal designs for seven objects and a large number of weighings, *Linear and Multilinear Algebra*, 50 (2002) 61-74.

[Pu] F. Pukelsheim, *Optimal Design of Experiments*, Wiley, New York, 1993.

[PW] N. Pullman and N. Wormald, Regular graphs with prescribed odd girth, *Utilitas Math.*, 24 (1983) 243-251.

[Ri] T. Rivlin, *The Chebyshev Polynomials*, Wiley, New York, 1974.

[RW] R.W. Robinson and W.C. Wormald, Number of cubic graphs, *J. Graph Theory*, 7 (1983) 463-467.

[Sa1] H. Sachs, Über die Anzahlen von Bäumen, Wäldern und Kreisen gegebenen Typs in gegebenen Graphen, *Habilitationsschrift Univ. Halle, Math.-Nat. Fak.* (1963).

[Sa2] H. Sachs, Beziehungen zwischen den in einem Graphen enthaltenen Kreisen und seinem charakteristischen Polynom, *Publ. Math. Debrecen* 11 (1964) 119-134.

[St] F.W. Stevenson, *Projective Planes*, Freeman, San Francisco, 1972.

[vM] H. van Maldeghem, *Generalized Polygons*, Monographs in Mathematics Vol. 93, Birkhäuser Verlag, Basel, (1998)

[vLW] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

[Ya] F. Yates, Complex experiments, *J. Roy. Statist. Soc. Supp.* 2 (1935) 181-247.