Matching points with geometric objects: Combinatorial results

Bernardo M. Ábrego[1], Esther Arkin[2], Silvia Fernández-Merchant[1],
Ferran Hurtado[3], Mikio Kano[4], Joseph S. B. Mitchell[2], and Jorge Urrutia[5]

[1] Department of Mathematics, California State University, Northridge
[2] Department of Applied Mathematics and Statistics,
State University of New York at Stony Brook
[3] Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya
[4] Ibaraki University
[5] Instituto de Matemáticas, Universidad Nacional Autónoma de México

Abstract. Given a class $\mathcal{C}$ of geometric objects and a point set P, a $\mathcal{C}$-matching of P is a set $M = \{C_1, \ldots, C_k\}$ of elements of $\mathcal{C}$ such that every C_i contains exactly two elements of P. If all the elements of P belong to some C_i, M is called a *perfect matching*; if in addition all the elements of M are pairwise disjoint we say that this matching M is *strong*. In this paper we study the existence and characteristics of $\mathcal{C}$-matchings for point sets on the plane when $\mathcal{C}$ is the set of circles or the set of isothetic squares on the plane.

1   Introduction

Let $\mathcal{C}$ be a class of geometric objects and let P be a point set with n elements $p_1, \ldots, p_n$ in general position, n even. A $\mathcal{C}$-matching of P is a set $M = \{C_1, \ldots, C_k\}$ of elements of $\mathcal{C}$, such that every C_i contains exactly two elements of P. If all the elements of P belong to some C_i, M is called a *perfect matching*. If in addition all the elements of M are pairwise disjoint we say that the matching M is *strong*.

If we define a graph $G_{\mathcal{C}}(P)$ in which the vertices are the elements of P, two of which are adjacent if there is an element of $\mathcal{C}$ containing them and no other element from P, a perfect matching in $G_{\mathcal{C}}(P)$ in the graph theory sense corresponds naturally with our definition of $G_{\mathcal{C}}(P)$-matchings.

If $\mathcal{C}$ is the set of line segments or the set of all isothetic rectangles, then we get a *segment-matching* or a *rectangle-matching*, respectively. If $\mathcal{C}$ is the set of disks on the plane, M will be called a *circle-matching*. If $\mathcal{C}$ is the set of all isothetic squares, M will be called a *square-matching*. Notice that these four classes of objects have in common the *shrinkability* property: if there is an object C' in the class that contains exactly two points p and q in P, then there is an object C'' in the class such that $C'' \subset C'$, p and q lie on the boundary of C'', and the relative interior of C'' is empty of points from P. In the case of rectangles we can even assume the points p and q to be opposite corners of C''.

It is easy to see that P always admits a strong segment-matching and a strong rectangle-matching, which in fact are respectively non-crossing matchings in the complete geometric graph induced by P (in the sense in which geometric graphs are defined in [4]) and in the rectangle of influence graph associated to P [3].

On the contrary, the situation is unclear for circles and squares, and gives rise to interesting problems, That is the topic of this paper, in which we study the existence of perfect and non perfect, strong and non strong matchings for point sets on the plane when $\mathcal{C}$ is the set of circles or the set of isothetic squares on the plane.

2 Matching with disks

In this section we study circle matchings. We show that a perfect circle matching is always possible, but that there are collections of points for which there is no perfect strong circle matching. We then give bounds on the size of the largest strong circle matching that any set P of n points would admit. In the last part of the section we consider the special case in which the point set P is in convex position.

2.1 Existence of Circle Matchings

First of all, notice that the fact that two points from P can be covered by a disk that contains no other point in P is equivalent to say that the two points are neighbors in the Delaunay triangulation of P, $DT(P)$. In other words, when $\mathcal{C}$ is the set of all circles on the plane, the graph $G_{\mathcal{C}}(P)$ is $DT(P)$. As a consequence a point set will admit a circle-matching if and only if the graph $DT(P)$ contains a perfect matching, which when P has an even number of points is always the case, as proved by Dillencourt in 1990 [2]. Therefore we get the following result, which is a direct consequence of Dillencourt's result:

Theorem 1. *Every point set with an even number of elements admits a circle-matching.*

Nevertheless a perfect strong circle-matching is not always possible as we show next. Consider a circle C with unit radius and a point set P with n elements $p_1, \ldots, p_n$, where $p_1 = a$ is the center of C and $p_2, \ldots, p_n$ are points evenly spaced on the boundary of C. The point a has to be matched with some point $b \in \{p_2, \ldots, p_n\}$; this forces that the rest of points are matched consecutively (see Figure 1), in particular the following and preceding neighbors of b on the boundary have to be matched using "large" circles that are pushed outside of C and overlap for n large enough. In fact, elementary trigonometric computations show that this happen exactly for $n \geq 74$.

We don't describe the details of the preceding construction, because the underlying basic ideas can we used for constructing an arbitrarily large set of points such that at most a certain fraction of the points can be strongly matched. More precisely, the following result holds:
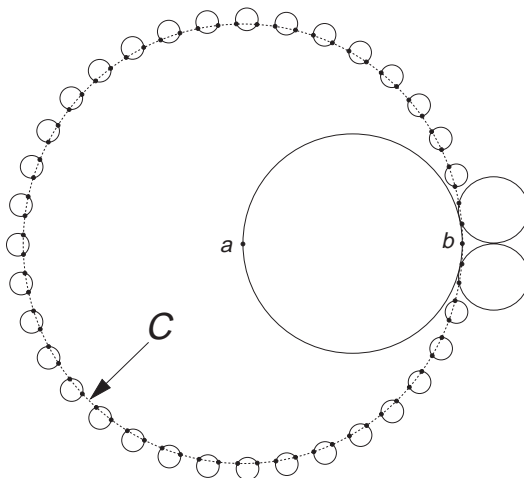
Fig. 1. The elements of a set S are $n-1$ points evenly distributed on C and the center of C. For $n \geq 74$ this point set does not admit any strong perfect circle-matching.

Theorem 2. *There is an n-element point set in the plane, where n can be arbitrarily large, such that at most a fraction $\frac{72}{73}n$ of its points can be strongly circle-matched.*

The proof of this result is omitted from this extended abstract, because it is very long and requires several technical lemmas.

2.2 Subsets that can be matched strongly

According to Theorem 2, not every point set P admits a strong circle-matching. Here we prove that, at least, we can always find a linear number of disjoint disks each one covering exactly two points from P:

Theorem 3. *For every P with n points in general position, there is a strong circle-matching using at least $2\lceil (n-1)/8 \rceil$ points of P.*

Let M be the minimum square-distance matching of P, that is M consists of $m = \lfloor n/2 \rfloor$ pairs of points $p_1q_1, p_2q_2, \ldots, p_mq_m$ where all p_is and q_is are different and the sum $\sum_{i=1}^{m} p_iq_i^2$ is minimum among all possible choices of the pairs p_iq_i. Let $\mathcal{C}$ be the diametral disks determined by the pairs p_iq_i in M. We denote by $D_i = DD(p_iq_i)$ the closed disk with diameter p_iq_i and by o_i the center of $DD(p_i, q_i)$. We first prove the following lemmas.

Lemma 1. *If $DD(ab), DD(cd) \in \mathcal{C}$ then $\{c, d\} \nsubseteq DD(ab)$.*

Proof. Suppose that $c, d \in DD(ab)$. Note that $\angle dcb + \angle bdc < \pi$, so we may assume that $\angle dcb < \pi/2$. Thus $bd^2 < cd^2 + bc^2$, and since $c \in DD(ab)$, then

$\angle bca \geq \pi/2$ and $bc^2 + ac^2 \leq ab^2$. Combining these inequalities we get $bd^2 + ac^2 < ab^2 + cd^2$ contradicting the minimality of M. $\qquad\square$

Lemma 2. *If $DD(ab), DD(cd) \in \mathcal{C}$ and p is in the intersection of the bounding circles of $DD(ab)$ and $DD(cd)$, then the triangles apb and dpc do not overlap.*

Proof. Suppose that $\triangle apb$ and $\triangle dpc$ overlap. Assume $\overrightarrow{pd}$ is between $\overrightarrow{pb}$ and $\overrightarrow{pa}$ as in Figure 2. Since $\overline{ab}$ and $\overline{cd}$ are diameters of their respective circles, then $\angle dpc = \angle apb = \pi/2$. So $\angle apd < \pi/2$ and $\angle bpc < \pi/2$. Then $ad^2 < pa^2 + pd^2$, $bc^2 < pb^2 + pc^2$, and

$$ad^2 + bc^2 < pa^2 + pb^2 + pc^2 + pd^2 = ab^2 + cd^2,$$

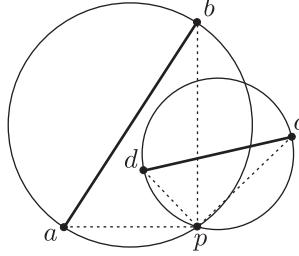which contradicts the minimality of M. $\qquad\square$

Fig. 2. Proof of Lemma 2

Lemma 3. *No three disks in $\mathcal{C}$ have a common intersection.*

Proof. Suppose $I = DD(p_1q_1) \cap DD(p_2q_2) \cap DD(p_3q_3) \neq \varnothing$. By Lemma 1, the boundary of I must contain sections of at least two of the bounding circles of $DD(p_1q_1)$, $DD(p_2q_2)$ and $DD(p_3q_3)$. Thus we may assume there is a point $p \in I$ such that p is in the intersection of the bounding circles of $DD(p_1q_1)$ and $DD(p_2q_2)$. By Lemma 2 the triangles $\triangle pp_1q_1$ and $\triangle pp_2q_2$ do not overlap. Now we consider three cases depending on the number of triangles that overlap with $\triangle pp_3q_3$.
Case 1. $\triangle pp_3q_3$ does not overlap with $\triangle pp_1q_1$ or $\triangle pp_2q_2$.

We may assume the relative order of the triangles pp_iq_i is as in Figure 3. Then, since $\angle q_1pp_1, \angle q_2pp_2, \angle q_3pp_3 \geq \pi/2$, we have that

$$\angle p_2pq_1 + \angle p_3pq_2 + \angle p_1pq_3 \leq \pi/2.$$

So all these angles are at most $\pi/2$ with at least two of them strictly acute (or zero). Thus

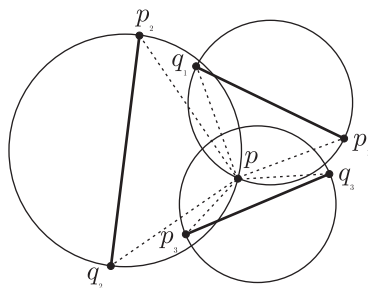$$q_1p_2^2 + q_2p_3^2 + q_3p_1^2 < pq_1^2 + pp_2^2 + pq_2^2 + pp_3^2 + pq_3^2 + pp_1^2.$$

Fig. 3. Proof of Case 1: no overlap

Also, since none of the angles $\angle q_1pp_1, \angle q_2pp_2, \angle q_3pp_3$ are acute then

$$pp_1^2 + pq_1^2 + pp_2^2 + pq_2^2 + pp_3^2 + pq_3^2 \leq p_1q_1^2 + p_2q_2^2 + p_3q_3^2.$$

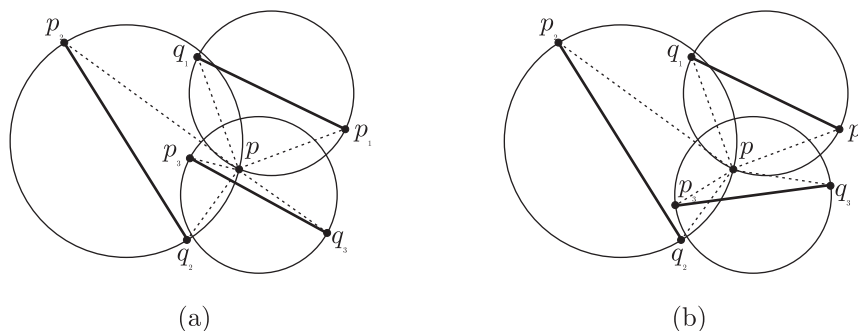Thus $q_1p_2^2 + q_2p_3^2 + q_3p_1^2 < p_1q_1^2 + p_2q_2^2 + p_3q_3^2$ which contradicts the minimality of M.

(a)   (b)

Fig. 4. Proof of Case 2: $\triangle pp_3q_3$ overlaps $\triangle pp_2q_2$

Case 2. $\triangle pp_3q_3$ overlaps with $\triangle pp_2q_2$ but not with $\triangle pp_1q_1$.

Assume $\overrightarrow{pp_3}$ is between $\overrightarrow{pp_2}$ and $\overrightarrow{pq_2}$. We may also assume that $\angle q_3pp_3 > \pi/2$, otherwise p is in the bounding circle of $DD(p_3q_3)$ and then by Lemma 2 $\triangle pp_3q_3$ and $\triangle pp_2q_2$ do not overlap. Since $\angle q_3pp_3 > \pi/2$ then $p_3q_3^2 > pp_3^2 + pq_3^2$.

If $\angle q_3pq_2 \leq \pi/2$ (Figure 4a) then, same as in the proof of Lemma 2, $q_2q_3^2 \leq pq_2^2 + pq_3^2$, $p_2p_3^2 \leq pp_2^2 + pp_3^2$, and then

$$q_2q_3^2 + p_2p_3^2 \leq pp_2^2 + pq_2^2 + pp_3^2 + pq_3^2 < p_2q_2^2 + p_3q_3^2,$$

which contradicts the minimality of M.

If $\angle q_3pq_2 > \pi/2$ (Figure 4b) then $\angle p_1pq_3 + \angle p_2pq_1 < \pi/2$. Thus $\angle p_1pq_3$, $\angle p_2pq_1 < \pi/2$ and then $p_1q_3^2 < pp_1^2 + pq_3^2$ and $p_2q_1^2 < pp_2^2 + pq_1^2$. Also, since $\overrightarrow{pp_3}$ is between $\overrightarrow{pp_2}$ and $\overrightarrow{pq_2}$, then $\angle q_2pq_3 < \pi/2$ and $q_2p_3^2 < pq_2^2 + pp_3^2$. Putting all these together we get

$$p_1q_3^2 + p_2q_1^2 + q_2p_3^2 < pp_1^2 + pq_1^2 + pp_2^2 + pq_2^2 + pp_3^2 + pq_3^2.$$

Moreover $pp_1^2 + pq_1^2 = p_1q_1^2$, $pp_2^2 + pq_2^2 = p_2q_2^2$, and $pp_3^2 + pq_3^2 < p_3q_3^2$. Thus

$$p_1q_3^2 + p_2q_1^2 + q_2p_3^2 < p_1q_1^2 + p_2q_2^2 + p_3q_3^2$$

which contradicts the minimality of M.

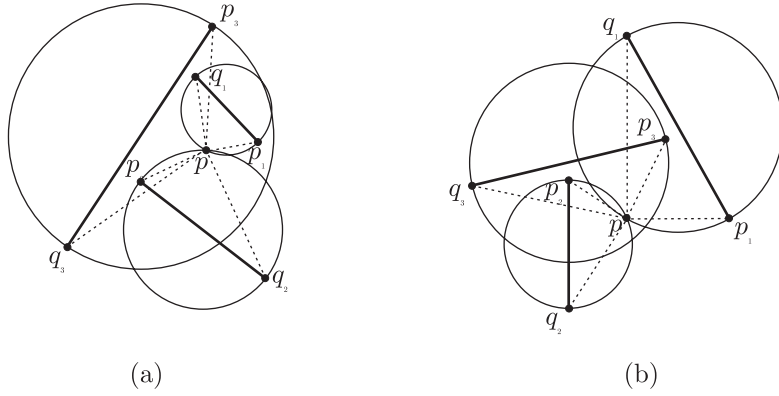

(a)   (b)

Fig. 5. Proof of case 3: $\triangle pp_3q_3$ overlaps $\triangle pp_2q_2$ and $\triangle pp_1q_1$.

Case 3. $\triangle pp_3q_3$ overlaps $\triangle pp_2q_2$ and $\triangle pp_1q_1$.

We may assume $\overrightarrow{pp_3}$ and $\overrightarrow{pq_3}$ are between $\overrightarrow{pp_1}, \overrightarrow{pq_1}$ and $\overrightarrow{pp_2}, \overrightarrow{pq_2}$ respectively (Figure 5). Again by Lemma 2 we may assume that $\angle q_3pp_3 > \pi/2$ and $p_3q_3^2 > pp_3^2 + pq_3^2$.

If $\angle p_1pq_2 \leq \pi/2$ (see Figure 5a) then $p_1q_2^2 \leq pp_1^2 + pq_2^2$. From the locations of p_3 and q_3 we have that $\angle q_1pp_3$, $\angle q_3pp_2 < \pi/2$ and $p_3q_1^2 < pq_1^2 + pp_3^2$, $p_2q_3^2 < pp_2^2 + pq_3^2$. Then

$$p_1q_2^2 + p_3q_1^2 + p_2q_3^2 < pp_1^2 + pq_1^2 + pp_2^2 + pq_2^2 + pp_3^2 + pq_3^2.$$

In addition $pp_1^2 + pq_1^2 = p_1q_1^2$, $pp_2^2 + pq_2^2 = p_2q_2^2$, and $pp_3^2 + pq_3^2 < p_3q_3^2$. Thus

$$p_1q_2^2 + p_3q_1^2 + p_2q_3^2 < p_1q_1^2 + p_2q_2^2 + p_3q_3^2.$$

If $\angle p_1pq_2 > \pi/2$ then, (see Figure 5b) in a similar way, we get

$$p_2q_1^2 + p_1p_3^2 + q_2q_3^2 < p_1q_1^2 + p_2q_2^2 + p_3q_3^2.$$

In both cases we contradict the minimality of M.   □

Lemma 4. *If $D_1, D_2, D_3, D_4 \in \mathcal{C}$ with $D_1 \cap D_2 \neq \varnothing$ and $D_3 \cap D_4 \neq \varnothing$ then the segments $o_1 o_2$ and $o_3 o_4$ do not intersect.*

Proof. Suppose $\overline{o_1 o_2}$ and $\overline{o_3 o_4}$ intersect. Let x be the intersection of the two segments, $p \in D_1 \cap D_2 \cap \overline{o_1 o_2}$, and $q \in D_3 \cap D_4 \cap \overline{o_3 o_4}$. Assume $p \in \overline{x o_2}$ and $q \in \overline{x o_4}$. By the Triangle Inequality $o_1 q \leq o_1 x + x q$ and $o_3 p \leq o_3 x + x p$, then

$$o_1 q + o_3 p \leq o_1 x + x p + o_3 x + x q = o_1 p + o_3 q.$$

Thus either $o_1 q \leq o_1 p$ or $o_3 p \leq o_3 q$, which implies that either $q \in D_1$ or $p \in D_3$. This is a contradiction to Lemma 3, since either $q \in D_1 \cap D_3 \cap D_4$ or $p \in D_1 \cap D_2 \cap D_3$. $\qquad\square$

Proof of Theorem 3. Let G be a graph with vertex set the centers of the disks in $\mathcal{C}$, with two vertices connected by an edge if the corresponding disks intersect. By the last lemma, G is a planar graph. Then by the Four Color Theorem the maximum independent set of G has at least $\lceil m/4 \rceil = \lceil \lfloor n/2 \rfloor /4 \rceil = \lceil (n-1)/8 \rceil$ vertices. Thus the corresponding diametral disks are pairwise disjoint. Therefore P has a circle-matching using at least $2\lceil (n-1)/8 \rceil$ points. It may happen that these diametral circles have points of P in their interior. However it is always possible to find a circle inside one of these diametral circles, containing only two points of P.

2.3 Convex position

When we have n points on a line, with n even, it is obvious that a strong perfect matching with disks is always possible, as we can simply take the diametral circles defined by consecutive pairs. As a consequence a strong perfect matching is also always possible when we are given any set P of n points lying on a circle C: using an inversion with center at any point in $C \setminus P$ the images of all points from P become collinear and admit a matching, which applying again the same inversion gives the desired matching (because inversions are involutive and apply circles that don't pass through the center of inversion into circles).

This may suggest that a similar result would hold for any set of points in convex position, but this is not the case as we show next using the same kind of arguments.

Let Q be the point set shown in Figure 1, consisting of the center a of a circle C, and 73 points additional points evenly distributed on C; as commented, Q does not admit a strong perfect circle-matching.

Let P be the point set obtained from Q by applying any inversion with center at some point $p \in C$ which does not belong to Q; the point set P does not admit a strong perfect circle matching. Notice that all the points in P with the exception of the image of a lie on a line. Applying an infinitesimal perturbation to the elements of P in such a way they remain in convex position but no three are collinear produces a point set P' in convex position for which no strong perfect circle-matching exists, because the inverse set Q' is an infinitesimal perturbation

of Q and therefore does not admit a strong perfect circle-matching. Therefore we have proved the following result:

Proposition 1. *There are point sets in convex position in the plane that cannot be strongly circle-matched.*

3 Isothetic squares

In this section we consider the following variation to our geometric matching problem. Let P be a set of $2n$ points in general position on the plane. As in the previous section we define a graph $G(P)$ in which the points are the vertices of $G(P)$ two of which are adjacent if there is an isothetic square containing them that does not contain another element of P.

3.1 Existence of square-matchings

We show here that P always admits a square-matching. We prove first:

Lemma 5. $G(P)$ *is planar.*

Proof. Suppose that q_i is adjacent to q_j, and q_k to q_l and that the segments joining q_i to q_j and q_i to q_j intersect. Let $R_{i,j}$(resp. $R_{k,l}$) be the smallest isothetic rectangle containing q_i an q_j (resp. q_k and q_l). It is straightforward to see that either $R_{i,j}$ contains q_k or q_l or $R_{k,l}$ contains q_i or q_j. In the first case q_k and q_l cannot be adjacent in $G(P)$. In the second q_i and q_j are not adjacent in $G(P)$ which in both cases produce a contradiction. $\square$

Let C be a square that contains all the elements of P in its interior, and $P^{'}$ be the point set obtained by adding to P the vertices of C. Let G be the graph obtained from $G(P')$ by adding an extra point p_∞ adjacent to the vertices of C. We are going to see that G is 4-connected; before that we prove a technical lemma.

Lemma 6. *Let S be a point set containing the origin O and a point p from the first quadrant, such that all the others points in S lie in the interior of the rectangle R with corners at O and p. Then there is path in $G(S)$ from O to p such that every two consecutive vertices can be covered by an isothetic square contained in R, empty of any other point from S.*

Proof. The proof is by induction on $|S|$. If $|S| = 2$ the result is obvious. If $|S| > 2$ we grow homothetically from O a square with bottom left corner at O until a first point q from S, different from O, is found. This square is contained in R and gives an edge in $G(S)$ between O and q; now we can apply induction to the points from S covered by the rectangle with q and p as opposite corners. $\square$

Obviously the preceding result can be rephrased for any of the four quadrants to any point taken as origin. We are now ready for proving the following result:

Lemma 7. *G is 4-connected.*

Proof. Let us see that the graph G' resulting from the removal of any three vertices from G is connected.

Suppose first that none of the suppressed vertices is p_∞ and let us see that p_∞ can be reached from any vertex of G'. If a vertex $v \in G'$ is a corner of C, then it is adjacent to p_∞. If v is not such a corner, consider the four quadrants it defines. In at least one of them no vertex from G has been suppressed, then we can apply Lemma 6 to this quadrant and obtain a path in G' from v to a surviving corner of C; from there we arrive to p_∞.

If we suppress from G two points from P and p_∞, then G' contains the 4-cycle given by the corners of C. From any vertex $v \in P$ in G' we can reach one of these corners (and therefore any of them), because in at least two of the quadrants relative to v no vertex has been removed.

The cases in which p_∞ and one or two corners of C are suppresses are handled similarly. □
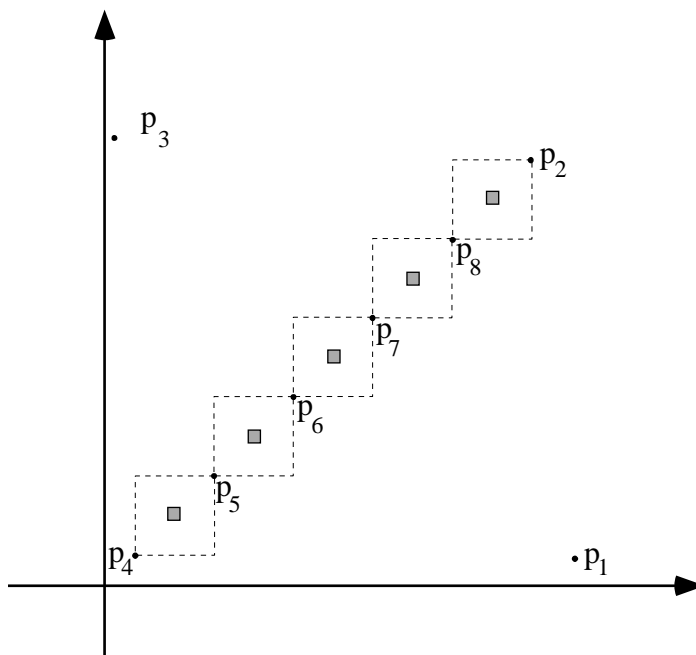

Fig. 6. Final step for the existence of square-matchings.

As it is clear that G is planar, it now follows using a classic result of Tutte [7] that G is Hamiltonian. This almost proves our result, since the removal of p_∞ from G results in a graph that has a Hamiltonian path. Using this path, we

can now obtain a perfect matching in $G(P')$. A small problem remains to find a matching in $G(P)$, namely the elements of P matched to the corners of C.

To solve this difficulty we proceed in a way similar to that use in [1]. Consider the five shaded squares and eight points $p_1, \ldots, p_8$ (represented by small circles) as shown in Figure 6. Within each of the shaded squares place a copy of P, and let P'' be the point set containing the points of the five copies of P plus $p_1, \ldots, p_8$. Consider the graph $G(P'')$ and add to it a vertex p_∞ adjacent to p_1, p_2, p_3, p_4. Once more $G(P'')$ is planar and four connected, and by Tutte's Theorem Hamiltonian. The removal of p_∞ gives a Hamiltonian path w in the resulting graph, with extremes in the set $\{p_1, p_2, p_3, p_4\}$. At least one of the five copies of P does not contain any neighbor of p_1 or p_3 in the path, because these two vertices have altogether at most four neighbors. Suppose, for example, that that is the case of the copy inside the box between p_5 and p_6; as the path has to arrive to the points inside the box and leave the set, and this can only be done through p_5 and p_6, the points inside the box have to be completely visited by the path w before leaving the box.

In this way we have obtained a Hamiltonian path in $G(P)$, which gives a perfect matching in $G(P)$, and thus we have proved:

Theorem 4. *P has a perfect square-matching.*

3.2 Subsets that can be strongly square-matched

We show first a family of 12 points that allows no perfect strong square-matching. Consider the point set with twelve points shown in Figure 7: there are four extreme points, labeled $p_1, ..., p_4$ and eight more points which are very close to the midpoints of an auxiliary dashed square as shown in the same figure; four of them, q_1, q_3, q_5, q_7, are *internal*, while four of them, q_2, q_4, q_6, q_8, are *external*. The points can all be drawn in general position, with no two on a common vertical/horizontal line.

Notice that no pair can be matched inside $\{p_1, ..., p_4\}$; therefore two pairs have to appear in any matching by picking points from $\{q_1, q_2, ..., q_8\}$. No two external points can be matched, and matching a close pair like q_1, q_2 would leave an extreme point (p_1 in the example) without any partner for a matching. Distributing the four internal points into two matched pairs produces always overlapping rectangles.

All this leaves only two possibilities for two pairs taken from $\{q_1, q_2, ..., q_8\}$: either using three internal points and one external point, or two internal points and two external points. In the first case the situation must be as in Figure 7, left, where q_1 is matched to q_7 and q_4 to q_5; this forces (p_3, q_6) and (p_4, q_8) to be matched pairs and causes overlap. In the second case the situation must be as in Figure 7, right, where q_2 is matched to q_3 and q_6 to q_7; this forces (p_1, q_1) and (p_4, q_8) to be matched pairs and causes overlap. This concludes the proof.

We now prove that the preceding result can be used to construct arbitrarily large sets which do not admit perfect strong square-matchings:
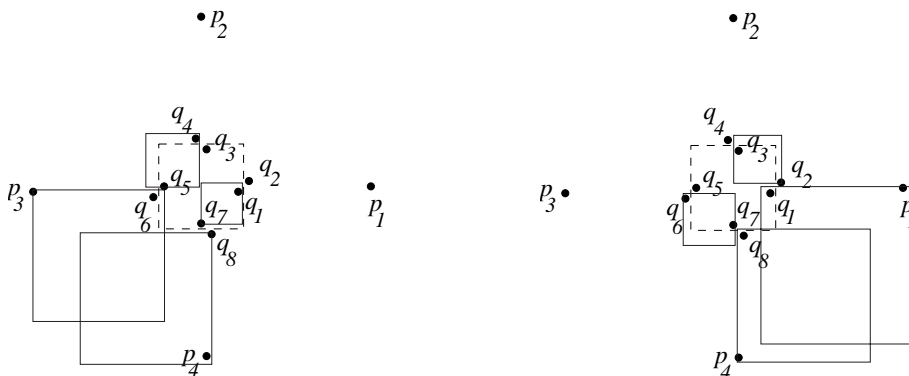
Fig. 7. Twelve points that do not admit a perfect strong square-matching

Proposition 2. *There are sets with $13m$ points such that any strong square matching of them contains at most $6m$ pairs of matched points.*

Proof. Take the line $y = x$, and consider the points with coordinates (i, i), $i = 1, ..., 2m + 1$, n even. For $i = 2i + 1$, $i = 0, ..., m - 1$ proceed as follows: Take an ϵ region around the point $(2i + 1, 2i + 1)$ and insert a copy of the 12 point configuration P_{12} (scaled down of to fit within this ϵ neighborhood). The remaining points (i, i) stay as singletons. Let P be the point set containing all these $12m + m$ points, and let M be a strong square-matching of P. See Figure 8

Observe that the 12 point set close to the point (1,1) cannot be matched within themselves. Then M matches at most 10 of these points. This leaves two points of pending. One of these points can be matched with point (2.2). The remaining point cannot be square matched with any point in P. In a similar way one of the points in the ϵ neighborhood of $(2i + 1, 2i + 1)$ cannot be matched to any element of P. This leaves at least n elements of P unmatched in M. Our result follows. □

We determine next a lower bound on the number of points of a point set that can always be strongly square-matched.

Theorem 5. *For every P with n points in general position, there is a strong square-matching using at least $2\lceil \frac{n}{5} \rceil$ points of P.*

In fact, we prove a slightly stronger result, from which the preseding theorem is immediately derived:

Lemma 8. *Let S be a square that contains a point set P with at least two elements. Then it is always possible to find a strong square matching of P with $\lceil \frac{n}{5} \rceil$ elements.*
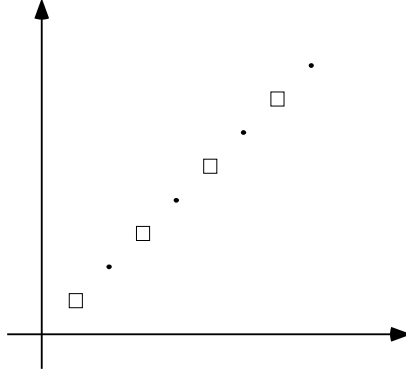
Fig. 8. Extending the twelve points counterexample for strong square-matchings

Proof. Our result is obviously true for $n = 2$. Suppose then that it is true for $n - 1$, and we now prove it for n, $n - 1 \geq 2$. Observe first that if $n = 5k + i$, $i = 2, 3, 4, 5$ then $\lceil \frac{n}{5} \rceil = \lceil \frac{n-1}{5} \rceil$, and by induction we are done. Suppose then that $n = 5k + 1$ for some k.

Partition S into for squares S_1, S_2, S_3, S_4 of equal size. Assume that each of them contains r_1, r_2, r_3, r_4 points respectively. If all r_i are greater than 2, or equal to zero, we are done since for any integers such that $r_1 + ... + r_4 = n$ we have:

$$\lceil \frac{r_1}{5} \rceil + ... + \lceil \frac{r_4}{5} \rceil \geq \lceil \frac{n}{5} \rceil$$

Suppose then that some of the r_i's are one. A case analysis follows.

Case 1: Three elements of the set $\{r_1, r_2, r_3, r_4\}$, say $r_2 = r_3 = r_4 = 1$ are equal to 1; $r_1 = 5(k - 1) + 3$.

Let $S_1^{'}$ be the smallest square, one of whose corners is p, that contains all the elements of P in S_1 but one, say p_1. Suppose w.l.o.g that p_1 lies below the horizontal line through the bottom edge of $S_1^{'}$. Then $S_1^{'}$ contains $5(k - 1) + 2$ points, and thus by induction we can find k disjoint squares in that square containing exactly two elements of P_n. It is easy to see that there is a square contained in $S - S_1^{'}$ that contains p_1 and the element of P_n in S_3. This square contains a square that contains exactly two elements of P. See Figure 9.

Case 2: Two elements of $\{r_1, r_2, r_3, r_4\}$ are equal to 1.

2a) Suppose that r_i and r_j are not 1. Observe that $r_i + r_j = 5k - 1$ and that $\lceil \frac{r_i}{5} \rceil + \lceil \frac{r_j}{5} \rceil \geq \lceil \frac{n-1}{5} \rceil$. If $\lceil \frac{r_i}{5} \rceil + \lceil \frac{r_j}{5} \rceil > \lceil \frac{n-1}{5} \rceil = k$ we are done. Suppose then that $\lceil \frac{r_i}{5} \rceil + \lceil \frac{r_j}{5} \rceil = \lceil \frac{n-1}{5} \rceil = k$; this happens only if one of them, say $r_i = 5r$ and the other element $r_j = 5s - 1$ for some r and s greater than or equal to zero.

Up to symmetry two cases arise:

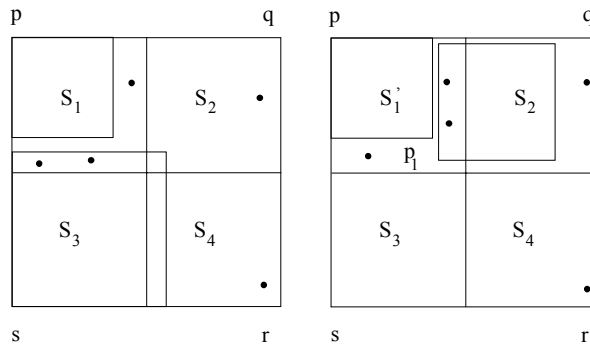2a1) $r_1 = 5r$ and $r_3 = 5s - 1$.

and

2a2) $r_1 = 5r$ and $r_4 = 5s - 1$

Fig. 9.

In the first case let S_1' be the smallest square contained in S_1 that contains all but three of the elements, say p_1, p_2 and p_3 of P in S_1, such that p is a vertex of S_1'. If two of these elements, say p_1 and p_2 are below the horizontal line through the lower horizontal edge of S_1', then there is a square S_3' contained in $S - S_1'$ that contains all the elements of P in S_3 and also contains p_1 and p_2, See Figure 10(a). Then by induction we can find in S_1' and S_3' $\lceil \frac{5r-3}{5} \rceil = r$ and $\lceil \frac{5s+1}{5} \rceil = s + 1$ disjoint squares, i.e. $r + s + 1 = k + 1$ disjoint squares contained in S each of which contains exactly two elements of P_n.
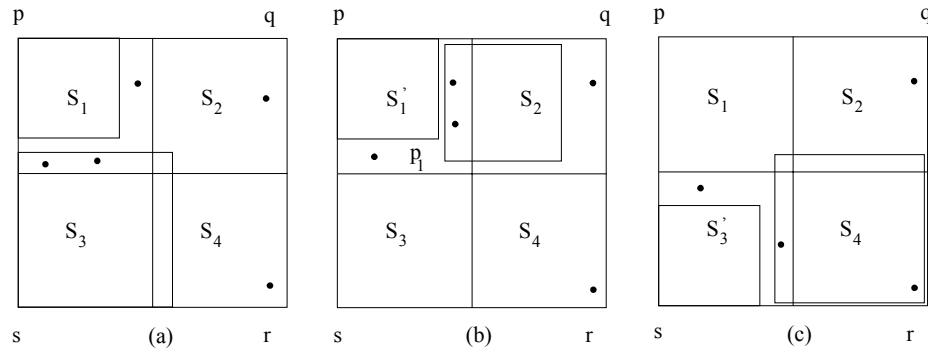

Fig. 10.

If no two elements of p_1, p_2 and p_3 lie below the horizontal through the lower horizontal edge of S_1', then there is a square contained in $S_1 \cup S_2 - S_1'$ that contains two of these elements. Applying induction to the elements of P in S_1',

the elements of P in S_3 and the square we just obtained prove our result. See Figure 10(b).

If $r = 0$, ad thus $s > 0$ choose S_3' such that it contains all but two points of P_n in S_3. If two points in S_3 lie above line containing the top edge of S_3' or to the right of the line L containing the rightmost vertical edge of S_3', an analysis similar to the one above follows. Suppose then that there is exactly one point in S_3 to the right of L. Then S_3' contains $5s - 3 \geq 2$ points, and there is a square contained in S containing the point of P_n in S_4. See Figure refcase2(c). By induction on the number of elements in S_3', and using the last square we obtained our result follows.

Case 2a2) can be solved in a similar way.

The remaining case, when only one of $\{r_1, r_2, r_3, r_4\}$ is 1 can be solved in a similar way to the previous cases. For example the case when only $r_4 = 1$, (in which case r_1, r_2 and r_3 are multiples of 5) $r_1 \neq 0$, and $r_2 = 0$ is solved almost the same way as case 2a1). We leave the details to the reader. $\qquad\square$

3.3 A perfect strong square-matching for the convex case

When several points may have the same x-coordinate or the same y-coordinate, a perfect strong matching is not always possible, as for example happens when we try to match point p in Figure 11 to any point on the line.
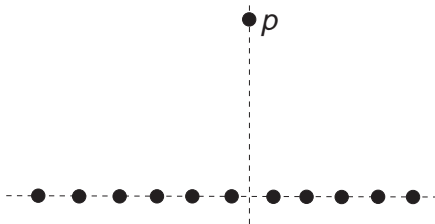

Fig. 11. Point p cannot be matched

Nevertheless, we can prove that in convex position, without repeated coordinates, a perfect strong matching always exists:

Theorem 6. *Any set of points in the plane in convex position with an even number of elements and such that no two points are in the same vertical or horizontal line, admits a perfect strong square-matching.*

The proof of this result is omitted from this extended abstract, because it is very long and requires several technical lemmas.

4 Concluding remarks

We have proved in this paper that (weak) matchings with circles and isothetic squares are always possible. It is natural to ask which other classes of convex objects would enjoy the same property, and try to characterize them. On the computational side, there are also decision and construction problems that are very interesting. These issues are the main lines of our ongoing research on the topic.

Acknowledgments

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