POINT-SETS IN GENERAL POSITION WITH MANY SIMILAR COPIES OF A PATTERN

BERNARDO M. ÁBREGO AND SILVIA FERNÁNDEZ-MERCHANT

Department of Mathematics
California State University, Northridge
18111 Nordhoff St, Northridge, CA 91330-8313.
email:{bernardo.abrego,silvia.fernandez}@csun.edu

ABSTRACT. For every pattern P, consisting of a finite set of points in the plane, $S'_P(n)$ is defined as the largest number of similar copies of P among sets of n points in the plane without 3 points on a line. A general construction, based on iterated Minkovski sums, is used to obtain new lower bounds for $S'_P(n)$ when P is an arbitrary pattern. Improved bounds are obtained when P is a triangle or a regular polygon with few sides.

1. INTRODUCTION

Sets A and B in the plane are *similar*, denoted by $A \sim B$, if there is an orientation-preserving isometry followed by a dilation that takes A to B. Identifying the plane with $\mathbb{C}$, the set of complex numbers, $A \sim B$ if there are complex numbers w and $z \neq 0$ such that $B = zA + w$. Here, $zA = \{za : a \in A\}$ and $A + w = \{a + w : a \in A\}$.

Consider a finite set of points P in the plane, $|P| \geq 3$. We refer to P as a *pattern* (P is usually fixed). For any finite set of points Q, we define $S_P(Q)$ to be the number of similar copies of P contained in Q. More precisely,

$$S_P(Q) = |\{P' \subseteq Q : P' \sim P\}|.$$

The main goal of this paper is to explicitly construct point sets Q in *general position* with a large number of similar copies of the pattern P, that is, with large $S_P(Q)$. By general position we mean to forbid

2000 *Mathematics Subject Classification.* Primary 52C10, Secondary 05C35.

Key words and phrases. Similar copy, pattern, geometric pattern, general position, collinear points, Minkovski Sum.

Some preliminary results appeared in [2].

triples of collinear points. In a forthcoming paper, we also consider the restriction of allowing at most m points on a line, $m \geq 3$, and the stronger restriction of not allowing collinear points or parallelograms in Q.

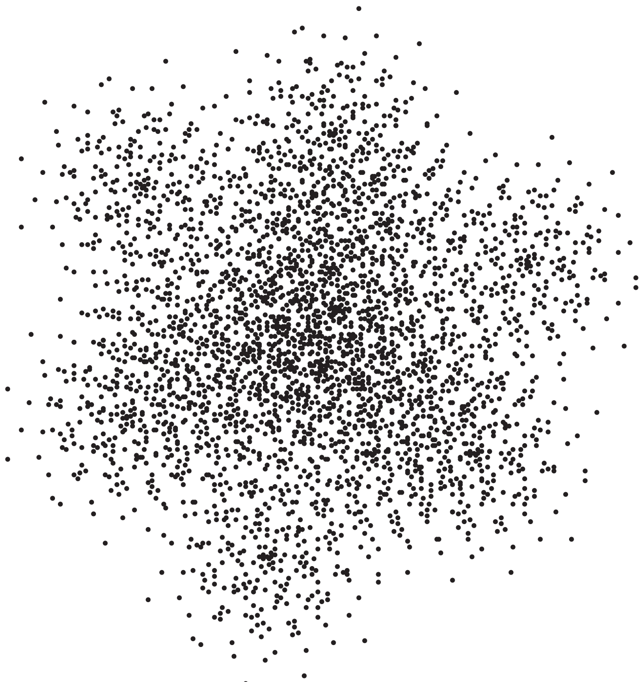

FIGURE 1. Point-set in general position with many triples spanning equilateral triangles.

To explain the motivation of this paper, we first turn to the original problem. Erdős and Purdy [8]-[10] posed the problem of maximizing the number of similar copies of P contained in a set of n points in the plane. That is, to determine the function

$$S_P(n) = \max_{|Q|=n} S_P(Q),$$

where the maximum is taken over all point-sets $Q \subseteq \mathbb{C}$ with n points. Elekes and Erdős [6] noted that $S_P(n) \leq n(n-1)$ for any pattern P. They gave a quadratic lower bound for $S_P(n)$ when $|P| = 3$ or when all the coordinates of the points in P are algebraic numbers. They also proved a slightly subquadratic lower bound for all other patterns P. Later, Laczkovich and Ruzsa [11] characterized precisely those patterns for which $S_P(n) = \Theta(n^2)$. However, the coefficient of the quadratic term is not known for any non-trivial pattern, not even for the simplest case of P an equilateral triangle [1]. Elekes and the authors [3] investigated the structural properties of the n-sets Q that achieve a quadratic number of similar copies of a pattern P. We proved that those sets contain large lattice-like structures, and therefore, many collinear points. In particular, the next result was obtained.

Theorem (Ábrego et al. [3])**.** *For every positive integer m and every real $c > 0$, there is a threshold function $N_0 = N_0(c, m)$ with the following property: if $n \geq N_0$ and Q is an n-set with $S_P(Q) \geq cn^2$, then Q has m points on a line forming an arithmetic progression.*

Thus having many points on a line is a required property to achieve $\Theta(n^2)$ similar copies of a pattern P. It is only natural to restrict the problem of maximizing $S_P(Q)$ over sets Q with a limited number of possible collinear points. For every natural number n, we restrict the maximum in $S_P(n)$ to n-sets in *general position*, that is, with no 3 points on a line. We denote this maximum by

$$S'_P(n) = \max\{S_P(Q) : |Q| = n \text{ and } Q \text{ in general position}\}.$$

We mention here that Erdős' unit distance problem [7] (arguably the most important problem in the area), has also been studied under the general position assumption of no 3 points on a line [10]. (See also [5, Section 5.1].)

The theorem above implies that $\lim_{n\to\infty} S'_P(n)/n^2 = 0$, i.e., $S'_P(n) = o(n^2)$. We believe that the true asymptotic value of $S'_P(n)$ is close to quadratic; however, prior to this work there were no lower bounds other than the trivial $S'_P(n) = \Omega(n)$. The rest of the paper is devoted to the construction of point-sets giving non-trivial lower bounds for $S'_P(n)$. In particular, when the pattern P is a triangle or a regular polygon.

2. Statement of results

The symmetries of the pattern P play an important role in the order of magnitude of our lower bound to the function $S'_P(n)$. Let us denote by $\mathrm{Iso}^+(P)$ the group of orientation-preserving isometries of the pattern P, also known as the *proper symmetry group* of P. Define the *index* of a point set A with respect to the pattern P, denoted by $i_P(A)$, as

$$i_P(A) = \frac{\log\left(\left|\text{Iso}^+(P)\right| S_P(A) + |A|\right)}{\log |A|}.$$

Observe that $1 \le i_P(A) \le 2$, because $\left|\text{Iso}^+(P)\right| S_P(A) \le |A|^2 - |A|$ as was noted by Elekes and Erdős, and moreover, $i_P(A) = 1$ if and only if $S_P(A) = 0$. Our main theorem gives a lower bound for $S'_P(n)$ using the index as the corresponding exponent. Its proof, presented in Section 3, is based on iterated Minkovski Sums.

Theorem 1. *For any A and P finite sets in the plane in general position, there is a constant $c = c(P, A)$ such that, for n large enough,*

$$S'_P(n) \ge cn^{i_P(A)}.$$

By using $A = P$, we conclude that the function $S'_P(n)$ is superlinear, that is, for any finite pattern P in general position, $S'_P(n) \ge \Omega(n^{\log(1+|P|)/\log|P|})$. So the key to obtaining good lower bounds for these functions is to begin with a set A with large index. For a general pattern P, we can marginally improve the last inequality by constructing a better initial set A. The proof of next theorem is in Section 4.1.

Theorem 2. *For any finite pattern P in general position and $|P| = k \ge 3$, there is a constant $c = c(P)$ such that, for n large enough,*

$$S'_P(n) \ge cn^{\log(k^2+k)/\log\left(k^2-k+1\right)}.$$

The following theorems summarize the lower bounds for $S'_P(n)$ obtained from the best known initial sets A for some specific patterns P. We concentrate on triangles and regular polygons. We often refer to a finite pattern as a geometric figure. For instance, when we say "let P be the equilateral triangle" we actually mean the set of vertices of an equilateral triangle.

Theorem 3. *Let $P = T$ be a triangle.*

- *For $T = \triangle$ equilateral, $S'_\triangle(n) \ge \Omega\left(n^{\log 102/\log 15}\right) \ge \Omega\left(n^{1.707}\right)$.*
- *For T isosceles, $S'_T(n) \ge \Omega\left(n^{\log 17/\log 8}\right) \ge \Omega\left(n^{1.362}\right)$.*
- *For T almost any scalene triangle, $S'_T(n) \ge \Omega\left(n^{\log 40/\log 14}\right) \ge \Omega\left(n^{1.397}\right)$. For all others, $S'_T(n) \ge \Omega\left(n^{\log 9/\log 5}\right) \ge \Omega\left(n^{1.365}\right)$.*

If P is a k-sided regular polygon, then $i_P(P) = \log(2k)/\log k$ and thus $S_P(n) \ge \Omega(n^{\log(2k)/\log k})$. If k is even, $4 \le k \le 10$ or if $k = 5$ we have the following improvement.

Theorem 4. *Let $P = R(k)$ be a regular k-gon. Then*

- $S'_{R(4)}(n) \geq \Omega\left(n^{\log 144/\log 24}\right) \geq \Omega\left(n^{1.563}\right)$,
- $S'_{R(6)}(n) \geq \Omega\left(n^{\log 528/\log 84}\right) \geq \Omega\left(n^{1.414}\right)$,
- $S'_{R(8)}(n) \geq \Omega\left(n^{\log 1312/\log 208}\right) \geq \Omega\left(n^{1.345}\right)$,
- $S'_{R(10)}(n) \geq \Omega\left(n^{\log 2640/\log 420}\right) \geq \Omega\left(n^{1.304}\right)$, *and*
- $S'_{R(5)}(n) \geq \Omega\left(n^{\log 264/\log 120}\right) \geq \Omega\left(n^{1.519}\right)$.

3. Proof of Theorem 1

Let P and A be sets in general position. With A as a base set, we recursively construct large sets in general position and with large number of similar copies of P. Our main tool is the *Minkowski Sum* of two sets $A, B \subseteq \mathbb{C}$, defined as the set $A+B = \{a+b : a \in A, b \in B\}$. First we have the following observation.

Proposition 1. *Let $P = \{p_1, p_2, \ldots, p_k\}$ and $Q = \{q_1, q_2, \ldots, q_k\}$ be sets with k elements. P is similar to Q, with p_j corresponding to q_j if and only if*

$$\frac{q_j - q_1}{q_2 - q_1} = \frac{p_j - p_1}{p_2 - p_1} \text{ for } j = 1, 2, \ldots, k.$$

Proof. If $P \sim Q$ with $q_j = zp_j + w$, where $z \neq 0$ and w are fixed complex numbers, then

$$\frac{q_j - q_1}{q_2 - q_1} = \frac{(zp_j + w) - (zp_1 + w)}{(zp_2 + w) - (zp_1 + w)} = \frac{p_j - p_1}{p_2 - p_1}.$$

Reciprocally, if $(q_j - q_1)/(q_2 - q_1) = (p_j - p_1)/(p_2 - p_1)$ for $1 \leq j \leq k$, then letting $z = (q_2 - q_1)/(p_2 - p_1)$ and $w = q_1 - zp_1$ we get that $q_j = zp_j + w$ and $z \neq 0$. $\square$

We now bound the number of copies of P in the sum $A + B$. To be concise, let $I = \left|\mathrm{Iso}^+(P)\right|$.

Lemma 1. *Let P be any finite pattern, and B and C finite sets such that $B + C$ has exactly $|B|\,|C|$ points. Then*

$$I \cdot S_P(B+C) + |B|\,|C| \geq \left(I \cdot S_P(B) + |B|\right)\left(I \cdot S_P(C) + |C|\right).$$

Proof. Suppose that $P = \{p_1, p_2, \ldots, p_k\}$ and let λ_j denote the ratio $(p_j - p_1)/(p_2 - p_1)$. Let $P_B = \{b_1, b_2, \ldots, b_k\} \subseteq B$ and $P_C =$

$\{c_1, c_2, \ldots, c_k\} \subseteq C$ be corresponding copies of P with $P \sim P_B \sim P_C$. Then by the previous proposition,

$$\frac{b_j - b_1}{b_2 - b_1} = \frac{c_j - c_1}{c_2 - c_1} = \frac{p_j - p_1}{p_2 - p_1} = \lambda_j.$$

Since $B + C$ has exactly $|B|\,|C|$ elements, then the relation $(b, c) \leftrightarrow b + c$ for $b \in B$, $c \in C$ is bijective. For any orientation-preserving isometry f of P_C (which uniquely corresponds to an element of $\mathrm{Iso}^+(P)$), consider the set $Q = \{q_j := b_j + f(c_j) : j = 1, 2, ..., k\} \subseteq B+C$. Since $(f(c_j) - f(c_1)) / (f(c_2) - f(c_1)) = (c_j - c_1) / (c_2 - c_1) = \lambda_j$, then

$$\begin{aligned}\frac{q_j - q_1}{q_2 - q_1} &= \frac{b_j - b_1 + f(c_j) - f(c_1)}{b_2 - b_1 + f(c_2) - f(c_1)} \\ &= \frac{\lambda_j (b_2 - b_1 + f(c_2) - f(c_1))}{b_2 - b_1 + f(c_2) - f(c_1)} = \lambda_j.\end{aligned}$$

That is, by the previous proposition, $Q \sim P$. Thus every similar copy P_B of P in B together with a similar copy P_C of P in C originate I distinct similar copies of P in $B + C$. We also have the 'liftings' of P in B and C. That is, similar copies of P of the form $(b, c_1), (b, c_2), \ldots, (b, c_k)$ or $(b_1, c), (b_2, c), \ldots, (b_k, c)$, with $b \in B$ and $c \in C$. All these copies of P in $B + C$ are different because $|B + C| = |B|\,|C|$. Therefore the number of similar copies of P in $B+C$ is at least $I \cdot S_P(B)\, S_P(C) + |B|\, S_P(C) + |C|\, S_P(B)$. In other words

$$\begin{aligned}I \cdot S_P(B + C) + |B|\,|C| \geq\ & I^2 \cdot S_P(B)\, S_P(C) + I \cdot |B|\, S_P(C) \\ & + I \cdot |C|\, S_P(B) + |B|\,|C| \\ & = (I \cdot S_P(B) + |B|)\,(I \cdot S_P(C) + |C|)\,.\square\end{aligned}$$

The next lemma allows us to preserve the maximum number of collinear points when we use the Minkovski Sum of two appropriate sets.

Lemma 2. *Let A and B be two sets with no m points on a line, $m \geq 3$. If $\mathcal{S}$ is the set of points $v \in \mathbb{C}$ for which $A + vB$ has less than $|A|\,|B|$ points, or m points on a line; then $\mathcal{S}$ has zero Lebesgue measure.*

For presentation purposes we defer the proof of this lemma and instead proceed to the proof of the theorem.

Proof of Theorem 1. Let $A_1 = A_1^* = A$ and suppose A_j and A_j^* have been defined. By Lemma 2 there is a set A_{j+1}, similar to A, such that $A_{j+1}^* := A_j^* + A_{j+1}$ is in general position and $|A_{j+1}^*| = |A_j^*||A_{j+1}|$. For every $j \geq 1$, $|A_j^*| = |A_{j-1}^*||A| = |A_{j-2}^*||A|^2 = \cdots = |A|^j$. Moreover, by Lemma 1, it follows that

$$\begin{aligned} I \cdot S_P\left(A_j^*\right) + \left|A_j^*\right| &= I \cdot S_P\left(A_{j-1}^* + A_j\right) + \left|A_{j-1}^*\right| \left|A_j\right| \\ &\geq \left(I \cdot S_P\left(A_{j-1}^*\right) + \left|A_{j-1}^*\right|\right)\left(I \cdot S_P\left(A\right) + \left|A\right|\right) \\ &\geq \left(I \cdot S_P\left(A\right) + \left|A\right|\right)^j . \end{aligned}$$

If $S_P(A) = 0$, then $i_P(A) = 1$ and the result is trivial. Assume $i_P(A) > 1$ and suppose $|A|^j \leq n < |A|^{j+1}$. The previous inequality yields

$$\begin{aligned} S'_P(n) \geq S_P(A_j^*) &\geq \frac{1}{I}((I \cdot S_P(A) + |A|)^j - |A|^j) \\ &\geq \frac{1}{I}\left(|A|^{j \cdot i_P(A)} - n\right) \geq \frac{1}{I}\left(\left(\frac{n}{|A|}\right)^{i_P(A)} - n\right) \\ &\geq c n^{i_P(A)}, \end{aligned}$$

for some constant $c = c(A, P)$ if n is large enough. For instance, $c = (2I\,|A|^{i_P(A)})^{-1}$ works whenever $n^{i_P(A)-1} \geq 2\,|A|^{i_P(A)}$. $\square$

Figure 1 shows a set A_3^* obtained from this procedure when $P = \triangle$ the equilateral triangle and A is the starting set with 15 points and 29 equilateral triangles constructed in Section 4.2.3. Finally, we present the proof of Lemma 2.

Proof of Lemma 2. We show that $\mathcal{S}$ is the union of a finite number of algebraic sets, all of them of real dimension at most one. This immediately implies that the Lebesgue measure of such a set is zero. For every $a \in A$ and $b \in B$, let $q(a, b) = a + vb$. Suppose that $q(a_1, b_1) = q(a_2, b_2)$ with $(a_1, b_1) \neq (a_2, b_2)$. Then $v(b_2 - b_1) + (a_2 - a_1) = 0$ and $b_1 \neq b_2$. Thus $v = -(a_2 - a_1)/(b_2 - b_1)$. Therefore there are at most $\binom{|A||B|}{2}$ values of v for which $A + vB$ has less than $|A|\,|B|$ points.

Now suppose that the set $\{q(a_j, b_j) : 1 \leq j \leq m\}$ consists of m points on a line, where $\{a_j\} \subseteq A$ and $\{b_j\} \subseteq B$. Then for $3 \leq j \leq m$, we have $q(a_j, b_j) - q(a_1, b_1) = \lambda_j\left(q(a_2, b_2) - q(a_1, b_1)\right)$, where

$\lambda_j \neq 0, 1$ is a real number. Thus for $3 \leq j \leq m$,

$$v\left(b_j - b_1 - \lambda_j\left(b_2 - b_1\right)\right) = \lambda_j\left(a_2 - a_1\right) - \left(a_j - a_1\right). \tag{1}$$

First assume all b_j are equal. Then all a_j are pairwise different, otherwise we would have less than m points initially. Moreover, Equation (1) implies that $(a_j - a_1)/(a_2 - a_1) = \lambda_j \in \mathbb{R}$ for all $3 \leq j \leq m$. But this contradicts the fact that there are no m points on a line in A. By possibly relabeling the points, we can now assume that $b_1 \neq b_2$. If $b_j - b_1 - \lambda_j\left(b_2 - b_1\right) \neq 0$ for some j, then

$$v = -\frac{\lambda_j\left(a_2 - a_1\right) - \left(a_j - a_1\right)}{\lambda_j\left(b_2 - b_1\right) - \left(b_j - b_1\right)}.$$

Möbius transformations send circles (or lines) to circles (or lines); thus the last equation, seen as a parametric equation on the real variable λ_j, represents a circle (or a line) in the plane. The remaining case is when

$$b_j - b_1 - \lambda_j\left(b_2 - b_1\right) = 0 \text{ for } 3 \leq j \leq m. \tag{2}$$

Since $\lambda_j \neq 0, 1$, then $b_j \neq b_1, b_2$. Suppose $b_j = b_k$ for $3 \leq j < k \leq m$, then $\lambda_j = \lambda_k$ and by (1) and (2) we deduce that $a_j = a_k$. This contradicts the fact that $q(a_j, b_j)$ and $q(a_k, b_k)$ are two different points. Therefore all the b_j are pairwise different, and then by (2) all the b_j are on a line. This is a contradiction since there are no m points on a line in B.

Therefore $\mathcal{S}$ is the union of a finite number of points and at most $\binom{|A||B|}{m}$ circles (or lines), and consequently it has zero Lebesgue measure. $\square$

4. Constructions of the initial sets

All the constructions of the initial sets A we provide have explicit coordinates so it is possible to calculate $S_P(A)$ via the following algorithm [4]: Fix two points $p_1, p_2 \in P$, for every ordered pair $(a_1, a_2) \in A \times A$ of distinct points consider the unique orientation-preserving similarity transformation f that maps $p_1 \mapsto a_1$ and $p_2 \mapsto a_2$. An explicit expression is $f(z) = \frac{a_1 - a_2}{p_1 - p_2} \cdot z + \frac{a_2 p_1 - a_1 p_2}{p_1 - p_2}$. Then verify whether $f(P) \subseteq A$. If N is the number of pairs (a_1, a_2) for which $f(P) \subseteq A$, then $S_P(A) = N / \left|\mathrm{Iso}^+(P)\right|$. The running time of this algorithm is $O(|P||A|^2 \log |A|)$. Likewise, it is possible to verify that no 3 points are on a line by simply checking the pairwise slopes of every triple of distinct points. We first present our construction for arbitrary patterns P.

4.1. **Arbitrary pattern P.**

Proof of Theorem 2. Let $z_0 \in \mathbb{C}\backslash P$ be an arbitrary point and let $p_1, p_2 \in P$ be two fixed points in P. For every $p \in P$, there is exactly one orientation-preserving similarity function f_p such that $p_1 \mapsto z_0$ and $p_2 \mapsto p$; indeed an explicit expression of such function is $f_p(z) = \frac{z_0 - p}{p_1 - p_2} \cdot z + \frac{pp_1 - z_0 p_2}{p_1 - p_2}$. Let A be the point set obtained by taking the image of P under every one of the functions f_p, that is, $A = \bigcup_{p \in P} f_p(P)$. For almost all z_0, except for a subset of real dimension 1, the set A does not have 3 points on a line and all the sets $f_p(P)\backslash\{z_0\}$ are pairwise disjoint, that is $|A| = 1 + k\,(k-1) = k^2 - k + 1$. This fact can be proved along the same lines as Lemma 2, we omit the details. By construction, each of the k sets $f_p(P)$ is

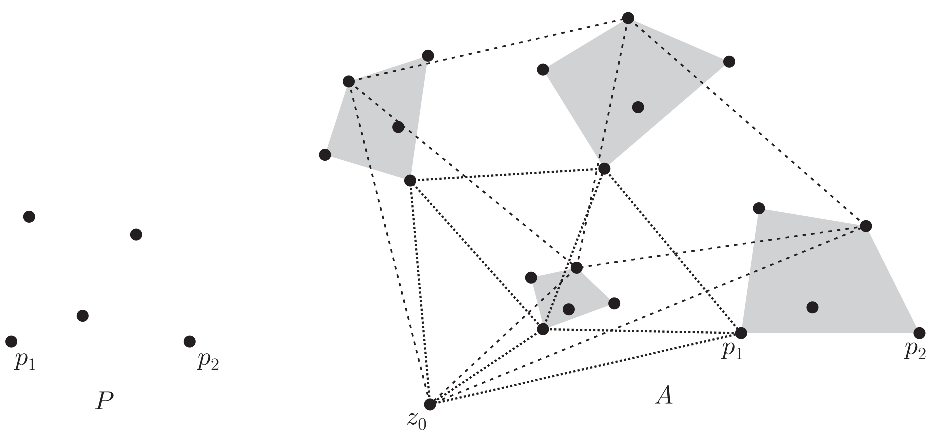

FIGURE 2. A has no m points on a line, $|A| = |P|^2 - |P| + 1$ points and $S_P(A) = 2\,|P| - 1$.

similar to P. For every $q \in P\backslash\{p_1\}$, the set

$$\{f_p(q) : p \in P\} = \frac{p_1 - q}{p_1 - p_2} P + \frac{z_0\,(q - p_2)}{p_1 - p_2} \tag{3}$$

is also similar to P and different from the previous copies of P we counted before. Thus $S_P(A) \geq 2k - 1$ and

$$i_P(A) \geq \frac{\log\left(S_P(A) + |A|\right)}{\log |A|} = \frac{\log\left(k^2 + k\right)}{\log\left(k^2 - k + 1\right)}.$$

The conclusion follows from Theorem 1. □

4.2. **Triangles.** The following table gives the currently best available initial set A for each pattern P in Theorem 3. That is, the set A with the largest index $i_P(A)$ known to date. The lower bound stated in Theorem 3 is then given by Theorem 1 applied to A. It is worth noting that our bound for scalene triangles is better than the one for (most) isosceles triangles. The intuitive reason for this is that it is harder to obtain better initial sets when the pattern has any type of symmetries. This difficulty is overtaken by the factor $\left|\mathrm{Iso}^+(P)\right| = 3$ when P is an equilateral triangle. We extend our comments in the concluding section.

		Best available A		
$P = T$ triangle	$\lvert\mathrm{Iso}^+(T)\rvert$	$\lvert A\rvert$	$S_T(A)$	$i_T(A)$
most scalene T	1	14	26	$\log 40/\log 14 > 1.397$
all scalene T	1	5	4	$\log 9/\log 5 > 1.365$
isosceles $T = T(\alpha)$ $\alpha \neq \pi/6, \pi/4, \pi/3$	1	8	9	$\log 17/\log 8 > 1.362$
$T(\pi/6)$	1	84	444	$\log 528/\log 84 > 1.414$
$T(\pi/4)$	1	24	120	$\log 144/\log 24 > 1.563$
equilateral $T(\pi/3)$	3	15	29	$\log 102/\log 15 = 1.707$

TABLE 1. Indices for the best initial sets when $P = T$ is a triangle.

4.2.1. *Scalene triangles.* We first give a construction for all scalene triangles. If the pattern $P = T$ consists of the points (complex numbers) $0, 1$, and $z \notin \mathbb{R}$, then the initial set $A_1 = \{0, 1, z, w = z - 1 + 1/z, wz\}$ is in general position for every scalene triangle T, see Figure 3(a). A_1 has 4 triangles similar to T: $(0, 1, z)$, $(z, w, 1)$, $(1, z, wz)$ and $(0, w, wz)$. For all z, but a subset of real dimension 1, the construction in Figure 3(b) is in general position and gives a better lower bound for $S'_T(n)$. The corresponding initial set is $A_2 = A \cup A'$ where $A = A_1 \cup zA_1 \cup \{wz^2/(z-1)\}$ and A' is the π-rotation of A about $wz/2$, that is $A' = -A + wz$. Because some points overlap, A_2 has only 14 points,

$$A_2 = \{0, 1, z, w, wz, z^2, wz^2, wz^2/(z-1), wz - 1,$$
$$wz - z, wz - w, 1 - z, wz(1-z), wz/(1-z)\}.$$

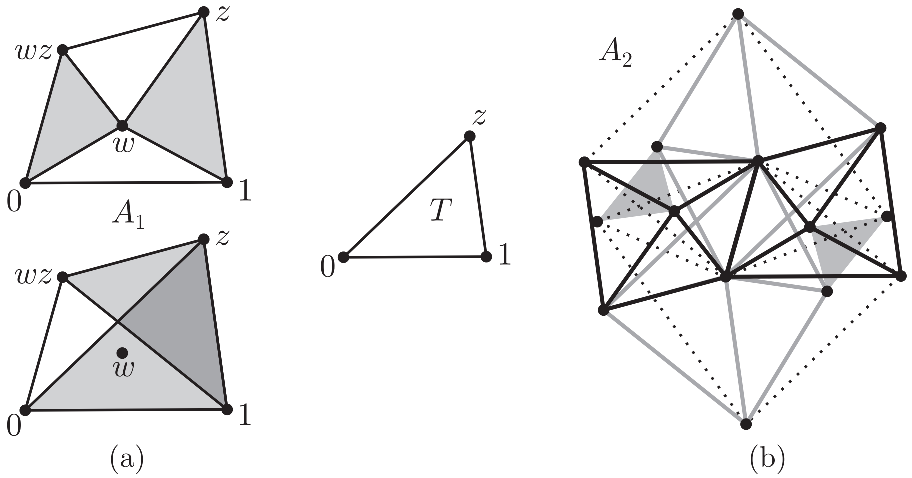

FIGURE 3. (a) 5-point set A_1 with $S_T(A_1) = 4$, (b) 14-point set A_2 with $S_T(A_2) = 26$.

Among the points in A_2, there are 6 similar copies of A_1 with no triangles similar to T in common: A_1, zA_1, $\frac{z}{1-z}(A_1 - wz)$, and their π-rotations about $wz/2$. In addition, the triangles $(w, wz(1-z), wz^2/(z-1))$ and $(wz^2, wz-w, wz/(1-z))$ are similar to T and are not contained in the 6 sets similar to A_1 mentioned before, so $S_T(A) \geq 26$.

4.2.2. *Isosceles triangles.* Let the pattern $P = T(\alpha)$ be an isosceles triangle with angles α, α, and $\pi - 2\alpha$, where $0 < \alpha < \pi/2$. We use as initial set one of the following two constructions, each with 8 points and 9 copies of $T(\alpha)$, i.e., $S_{T(\alpha)}(A) = 9$. There are three exceptions that are analyzed later in Section 4.3.3: $T(\pi/6)$, $T(\pi/4)$, and the equilateral triangle $T(\pi/3)$. Let $u = e^{2\alpha i}$ so that $T(\alpha) = \{0, 1, -u\}$. The first initial set is $A_1 = B_1 \cup \overline{B_1}$ where $B_1 = \{0, 1, u, 1+u, (2u+1)/(u+1)\}$ and $\overline{B_1}$ is the conjugate of B_1, i.e., $B_1 = \{\bar{b} : b \in B_1\}$. (See Figure 4(a).) This configuration is in general position as long as $\alpha \neq k\pi/12$, $k \in \mathbb{Z}$. It has 9 copies of $T(\alpha)$: $(0, 1, 1+u)$, $(1+u, u, 0)$, $(1, (2u+1)/(u+1), 1+u)$, $(1+1/u, (2u+1)/(u+1), u)$, their reflections about the real axis, and $(1/u, 1, u)$. The actual points in A_1 are

$$A_1 = \left\{0, 1, u, \frac{1}{u}, 1+u, 1+\frac{1}{u}, \frac{u+2}{u+1}, \frac{2u+1}{u+1}\right\}.$$

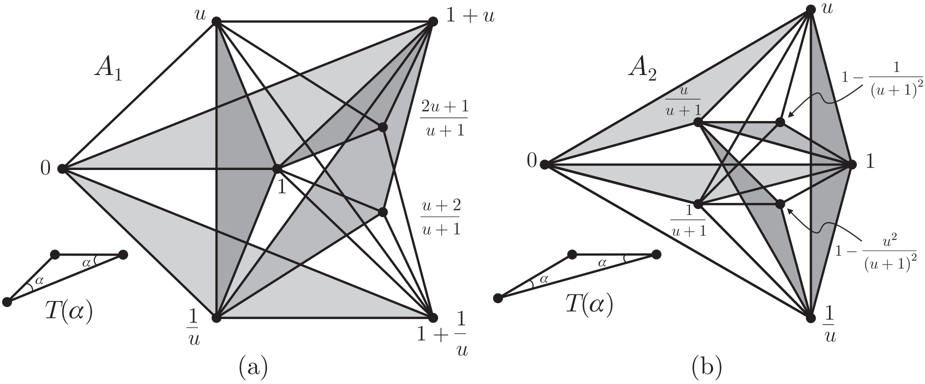

FIGURE 4. Sets A_i with $|A_i| = 8$ and $S_{T(\beta)}(A_i) = 9$. A_1 is in general position for $\alpha \neq \pi/12, \pi/6, \pi/4, \pi/3$, or $5\pi/12$, A_2 is in general position for $\alpha \neq \arccos\sqrt{(3+\sqrt{17})/8}$, $\pi/6$, $\pi/4$, or $\pi/3$.

The second initial set is $A_2 = B_2 \cup \overline{B_2}$ where $B_2 = \{0, 1, u, u/(u+1), 1-1/\left(u+1\right)^2\}$. (See Figure 4(b).) This set is in general position as long as $\alpha \neq \arccos\sqrt{(3+\sqrt{17})/8}, \pi/6, \pi/4$, or $\pi/3$. It has 9 copies of $T(\alpha)$: $(1, u/(u+1), 0)$, $(0, u/(u+1), u)$, $(1, 1-1/\left(u+1\right)^2, u/(u+1))$, $(1/(u+1), 1-1/\left(u+1\right)^2, u)$, their reflections about the real axis, and $(1/u, 1, u)$. The actual points in A_2 are

$$A_2 = \left\{0, 1, u, \frac{1}{u}, \frac{u}{u+1}, \frac{1}{u+1}, 1-\frac{1}{\left(u+1\right)^2}, 1-\frac{u^2}{\left(u+1\right)^2}\right\}.$$

4.2.3. *Equilateral triangle.* Let $z \in C$ and $\omega = e^{2i\pi/3}$ so that $\omega^2 + \omega + 1 = 0$. When the pattern $P = \triangle = \{1, \omega, \omega^2\}$ is the equilateral triangle, we use as initial set $A = B \cup \omega B \cup \omega^2 B$ where $B = \{1, -z\} \cup (-1 + zP)$. For all z but a 1-dimensional subset of $\mathbb{C}$, the set A is in general position and $|A| = 15$. The set $B_1 = \bigcup_{k=0}^{2} \omega^k(-1 + zP)$ is the Minkovski Sum $-P + zP$, thus by Lemma 1 there are at least 9 equilateral triangles in B_1. In addition $(1, \omega, \omega^2)$ and $(-z, -z\omega, -z\omega^2)$ are equilateral, and each of the points 1, ω, and ω^2 is incident to 6 more equilateral triangles: $(1, -\omega^2 + z, -\omega^2 z)$, $(1, -\omega^2 + z\omega, -z)$, $(1, -\omega^2 + z\omega^2, -z\omega)$, $(1, -\omega^2 z, -\omega + z\omega)$, $(1, -z, -\omega + z\omega^2)$, and $(1, -\omega z, -\omega + z)$, together with the $\pi/3$- and $2\pi/3$-rotations of these triangles about the origin. Thus $S_{\triangle}(A) \geq 29$. It can be checked that there are only 29 equilateral triangles in A.

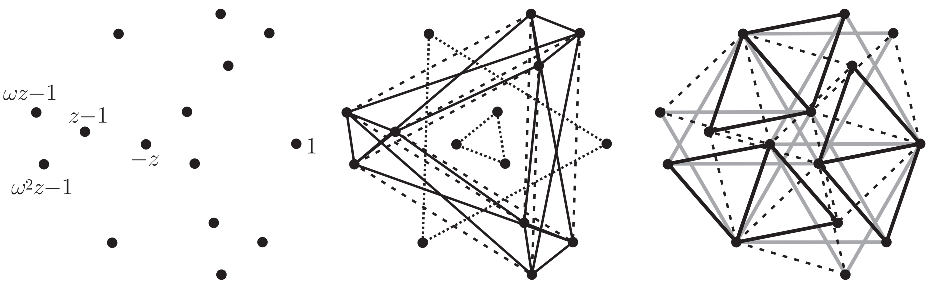

FIGURE 5. Initial set A with $|A| = 15$ and $S_{\triangle}(A) = 29$.

4.3. **Regular polygons.** As in the previous section, we construct initial sets for each k-regular polygon with $k \in \{4, 5, 6, 8, 10\}$. The following table gives the currently best available initial set A for each of these regular polygons.

		Best available A		
Polygon $R(k)$	$\lvert \mathrm{Iso}^+(R(k)) \rvert$	$\lvert A \rvert$	$S_{R(k)}(A)$	$i_{R(k)}(A)$
Square $= R(4)$	4	24	30	$\log 144/\log 24 > 1.563$
Hexagon $= R(6)$	6	74	84	$\log 528/\log 84 > 1.414$
Octagon $= R(8)$	8	208	138	$\log 1312/\log 208 > 1.345$
Decagon $= R(10)$	10	420	222	$\log 2640/\log 420 > 1.304$
Pentagon $= R(5)$	5	120	264	$\log 1440/\log 120 > 1.519$

TABLE 2. Indices for the best initial sets for k-regular polygons $R(k)$.

We first present the construction for the even-sided polygons and then the construction for the regular pentagon. Finally we explain how to use these initial sets for any pattern that is a subset of a regular polygon.

4.3.1. *Even sided regular polygons.* The following construction of the initial set A is in general position for all even k, however the index $i_{R(k)}(A)$ is only better than $i_{R(k)}(R(k))$ when $k \leq 10$. Let $\omega = e^{2\pi i/k}$, k even, and $z \in \mathbb{C}$ an arbitrary nonzero complex number. Suppose that the regular k-gon $R(k)$ is given by $R(k) = P = 1 + \omega + z\{\omega^j : 0 \leq j \leq k-1\}$. The reason why we translated the canonical regular polygon by $1+\omega$ and rotated and magnified it by z will become apparent soon. To construct our initial set A, we first follow the construction of Theorem 2 applied to P with $p_1 = 1+\omega+z$, $p_2 = 1+\omega+z\omega$, and $z_0 = 2$. We obtain a set A_1 consisting of z_0 and $k-1$ disjoint similar copies of P given by (3). That is, $A_1 = \{2\} \cup \bigcup_{j=1}^{k-1} B_j$ where

$$B_j = \frac{1-\omega^j}{1-\omega} P + \frac{2(\omega^j - \omega)}{1-\omega},\ 1 \le j \le k-1.$$

Note that there are exactly k similar copies of P with vertex $z_0 = 2$. Furthermore, $\omega^{k/2} = -1$ because k is even and thanks to the translation by $1+\omega$ in the definition of P, we have that

$$B_{k/2} = \left(\frac{2z}{1-\omega}\right)\left\{\omega^j : 0 \le j \le k-1\right\},$$

that is $B_{k/2}$ is a k-regular polygon centered at the origin. Now we add to the construction every rotation of A_1 by an integer multiple of $2\pi/k$. More precisely, we let

$$A = \bigcup_{l=0}^{k-1} \omega^l A_1. \tag{4}$$

Note that $B_{k/2}$ is a subset of all the sets $\omega^l A_1$. Because k is even,

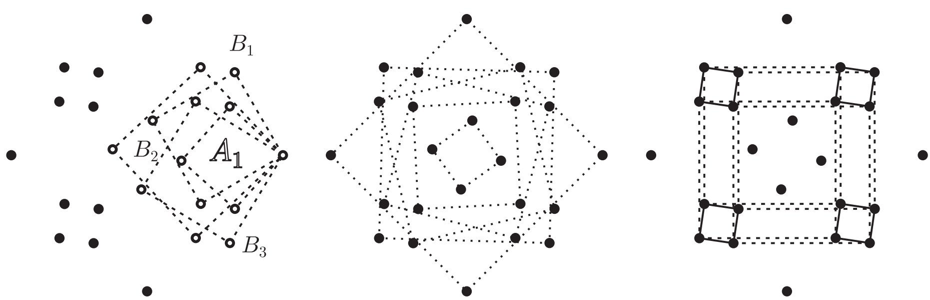

FIGURE 6. Initial set A for $R(4)$. $|A| = 24$ and $S_P(A) = 30$.

$-P = P - 2 - 2\omega$, thus for every $1 \le j \le k/2 - 1$,

$$\omega^j B_{k-j} = \frac{1-\omega^j}{1-\omega}(-P) + \frac{2(1-\omega^{j+1})}{1-\omega} = \frac{1-\omega^j}{1-\omega} P + \frac{2(\omega^j-\omega)}{1-\omega} = B_j.$$

For almost all $z \in \mathbb{C}$, except for a subset of real dimension 1, there are no 3 collinear points in A and also for every $j \ne k/2$ the sets B_j

and $\omega^l B_i$ are disjoint except for the pairs $(i,l) = (j,0)$ and $(i,l) = (k-j,j)$. It follows that every set of the form $\omega^l B_j$ with $j \neq k/2$ is a subset of exactly two terms in the union from Equation (4) and it is disjoint from the rest. Thus

$$
\begin{aligned}
|A| &= \left|B_{k/2}\right| + \left|\bigcup_{\substack{0\le l\le k-1\\ j\neq k/2}} \omega^l B_j\right| + \left|\left\{\omega^l z_0 : 0 \le l \le k-1\right\}\right| \\
&= k + \frac{1}{2}k\,(k-2) + k = \frac{k}{2}\left(k^2 - 2k + 4\right).
\end{aligned}
$$

To bound the number of k-regular polygons in A, first note that for each $0 \le l \le k-1$, there are at least k regular polygons with a vertex in $\omega^l z_0$ contained in $\omega^l A_1$ and all of these k^2 copies of P are different. For every $1 \le j \le k/2 - 1$ the set $\bigcup_{l=0}^{k-1} \omega^l B_j$ is the Minkovski Sum of two copies of P, namely

$$
\begin{aligned}
P_1 &= \left(1+\omega^j\right)\left\{\omega^l : 0 \le l \le k-1\right\} \text{ and} \\
P_2 &= \frac{1-\omega^j}{1-\omega} z\left\{\omega^l : 0 \le l \le k-1\right\},
\end{aligned}
$$

with exactly k^2 points. Thus, by Lemma 1, $S_P(\bigcup_{l=0}^{k-1} \omega^l B_j) = S_P(P_1 + P_2) \ge 3k$. Furthermore, all these $3k(k/2-1)$ regular polygons are distinct and also different from those previously counted. Finally, there are two extra polygons not yet counted, namely $B_{k/2}$ and $\left\{\omega^l z_0 : 0 \le l \le k-1\right\}$. Thus $S_P(A) \ge k^2 + 3k(k/2-1) + 2 = \frac{1}{2}(5k^2 - 6k + 4)$ and then

$$
\begin{aligned}
i_P(A) &\ge \frac{\log\left(\frac{k}{2}(5k^2-6k+4) + \frac{k}{2}\left(k^2-2k+4\right)\right)}{\log\left(\frac{k}{2}\left(k^2-2k+4\right)\right)} \\
&\ge \frac{\log\left(3k^3 - 4k^2 + 4k\right)}{\log\left(\frac{k}{2}\left(k^2-2k+4\right)\right)}.
\end{aligned}
$$

The conclusion follows by setting $k = 4, 6, 8,$ or 10.

4.3.2. *The regular pentagon.* Let $\omega = e^{2\pi i/5}$, for every $z \in C$ set $R(5) = P = z\{1, \omega, \omega^2, \omega^3, \omega^4\}$. Define

$$
A_1 = P + \frac{\sqrt{5}+3}{2},\ A_2 = \frac{\sqrt{5}+1}{2}\left(-P+1\right),
$$
$$
\text{and } A_3 = \left(\omega^2 - 1\right)\left\{z, \omega + 1\right\}.
$$

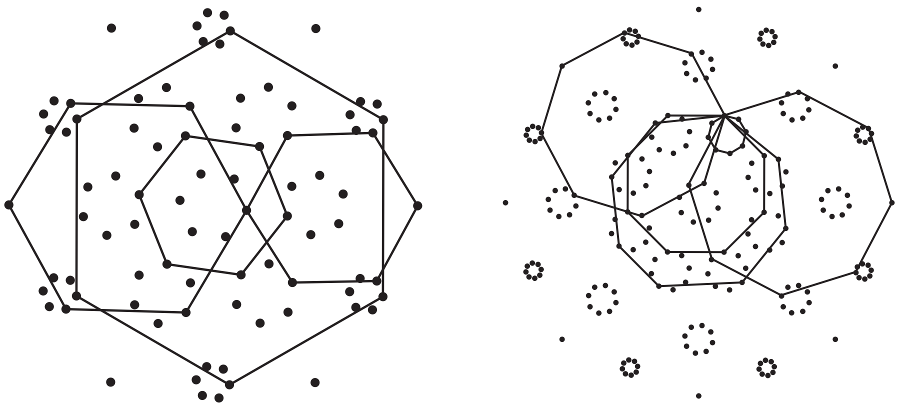

FIGURE 7. Best known initial sets for $P = R(6)$ and $P = R(8)$.

Now we consider all the $2\pi k/5$ rotations of these points, $0 \le k \le 4$, as well as their symmetrical points with respect to the origin. That is, we define $B = A_1 \cup (-A_1) \cup A_2 \cup (-A_2) \cup A_3 \cup (-A_3)$ and $A = \bigcup_{k=0}^{4} \omega^k B$. For almost all $z \in \mathbb{C}$, except for a subset of real dimension one, A has exactly 120 points and has no three points on a line. There are at least 264 regular pentagons with vertices in A: for each $j = 1, 2$, the set $\bigcup_{k=0}^{4} \omega^k (\pm A_j)$ is the Minkovski Sum of two regular pentagons, and thus by Lemma 1 each of these 4 sets has 15 regular pentagons, the point set $\bigcup_{k=0}^{4} \omega^k (\pm A_3)$ consists of two regular decagons so it contains 4 pentagons, finally each of the 20 points in $\bigcup_{k=0}^{4} \omega^k (\pm A_3)$ is incident to 10 more regular pentagons different from the ones previously counted (See Figure 8.) In fact every point in A is incident to exactly 11 regular pentagons and it turns out that the set A has an interesting set of automorphisms that preserve the regular pentagons.

4.3.3. *Subsets of regular polygons.* If P is a subset of a regular polygon R, then the constructions we have previously obtained for R would also be suitable for P. More precisely we have the following result.

Theorem 5. *Let R be a regular polygon and $P \subseteq R$ with $|P| \ge 3$. For every nonempty finite $A \subseteq \mathbb{C}$, we have that*

$$S_P(n) \ge \Omega\left(n^{i_R(A)}\right).$$

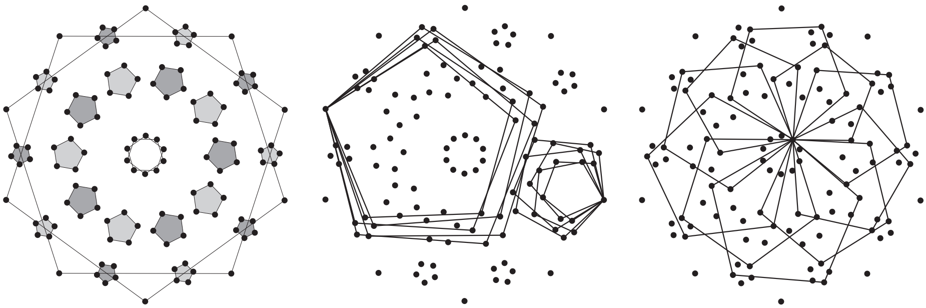

FIGURE 8. The best initial set A for the regular pentagon

Proof. Let $I = |Iso^+(P)|$. Because R is a regular polygon and $|P| \geq 3$, each similar copy of P in A is contained in at most one copy of R in A. Thus $S_P(A) \geq S_R(A) \cdot S_P(R)$. On the other hand, $S_P(R) = |R|/I$ and thus

$$I \cdot S_P(A) \geq I \cdot S_R(A) \cdot S_P(R) = |R|\, S_R(A).$$

Consequently

$$i_P(A) = \frac{\log\left(I \cdot S_P(A) + |A|\right)}{\log|A|} \geq \frac{\log\left(|R|\, S_R(A) + |A|\right)}{\log|A|} = i_R(A).$$

Finally, by Theorem 1, $S_P(n) \geq \Omega(n^{i_P(A)}) \geq \Omega(n^{i_R(A)})$. $\square$

As a direct consequence of this theorem, we take care of the isosceles triangles for which the construction in Section 4.2.2 yielded collinear triples. For the isosceles triangles $T(\alpha)$ with $\alpha = \pi/6$ or $\pi/4$ we have that $S_{T(\pi/6)}(n) \geq \Omega(n^{\log 528/\log 84})$ and $S_{T(\pi/4)}(n) \geq \Omega(n^{\log 144/\log 24})$, both of which exceed the bound in Theorem 3. Other point sets treated before can be improved this way as well. For instance $S_{T(\pi/5)}(n), S_{T(2\pi/5)}(n) \geq \Omega(n^{\log 1440/\log 120}) \geq \Omega(n^{1.519})$.

5. Conclusions and open problems

The main relevance of Theorem 1 is that it provides an effective tool to obtain better lower bounds for $S'_P(n)$. Indeed, any of the results for specific patterns in this paper, can be improved by finding initial sets with larger indices. For a general pattern P, Theorem 2 is only slightly better than the bound $\Omega(n^{\log(1+|P|)/\log|P|})$ which is obtained using $A = P$ as the initial set. There must be a way to construct a better initial set.

When the pattern $P = T$ is a triangle we obtained a considerably larger bound when T is equilateral. The reason behind this is the fact that there is a multiplying factor of $|\mathrm{Iso}^+(T)| = 3$ in the index of $i_T(A)$. We could not construct initial sets for arbitrary isosceles triangles that would improve the bound for scalene triangles. The mirror symmetry of the isosceles triangles became an obstacle when trying to construct sets with large indices. For instance, the set A_1 in Section 4.2.1 always yields collinear points when T is an isosceles triangle.

Problem 1. *For every isosceles triangle T, find a set A in general position such that $i_T(A) \geq \log 9/\log 5$.*

We also had some limitations to construct initial sets when $P = R(k)$ is a regular polygon. In this case, the index obtained using P itself as initial set is $\log(2k)/\log k$. We do not have a better initial set for odd $k \geq 7$ and in fact our construction of the initial set A for even-sided regular k-gons in Section 4.3.1 only gives an index $i_{R(k)}(A)$ better than $\log(2k)/\log k$ when $k < 12$. The constructions for the square and the pentagon have the property that every point belongs to the same number of regular polygons. Furthermore, their point–polygon incidence graphs are vertex transitive. It would be interesting to find such constructions for larger values of k.

Problem 2. *Let $R(k)$ be a regular k-gon. For every even $k \geq 12$ and odd $k \geq 7$ find a set A in general position such that $i_{R(k)}(A) \geq \log(2k)/\log k$.*

We are confident that there are some yet undiscovered methods for getting initial sets with larger indices. We would like to find such sets for some other classes of interesting geometric patterns.

Problem 3. *Find initial sets A with indices as large as possible when P is a right triangle, or a parallelogram, or a trapezoid.*

References

[1] B. M. Ábrego and S. Fernández-Merchant, On the maximum number of equilateral triangles I, Discrete and Computational Geometry 23 (2000), 129–135.

[2] B. M. Ábrego and S. Fernández-Merchant, On the maximum number of equilateral triangles II, DIMACS technical report TR99-47 (1999).

[3] B. M. Ábrego, S. Fernández-Merchant, and Gy. Elekes. Structural results for planar sets with many similar subsets, Combinatorica 24(4) (2004), 541–554.

[4] P. Brass, Combinatorial geometry problems in pattern recognition. Discrete and Computational Geometry, 28 (2002) 495–510.
[5] P. Brass, W. Moser, J. Pach, Research Problems in Discrete Geometry, Springer-Verlag, New York, 2005.
[6] G. Elekes and P. Erdős, Similar configurations and pseudogrids, in Intuitive Geometry, Colloquia Mathematica Societatis János Bolyai. North Holland, Amsterdam, (1994), 85–104.
[7] P. Erdős, On sets of distances of n points, Amer. Math. Monthly 53 (1946) 248-250.
[8] P. Erdős and G. Purdy. Some extremal problems in geometry. Journal of Combinatorial Theory 10 (1971), 246–252.
[9] P. Erdős and G. Purdy. Some extremal problems in geometry III. Proc. 6th Southeastern Conference in Combinatorics, Graph Theory and Comp. (1975), 291–308.
[10] P. Erdős and G. Purdy. Some extremal problems in geometry IV. Proc. 7th Southeastern Conference in Combinatorics, Graph Theory and Comp. (1976), 307–322.
[11] M. Laczkovich and I.Z.Ruzsa, The Number of Homothetic Subsets, The mathematics of Paul Erdős II, Springer Verlag (1997), 294–302.