Proximity Graphs inside Large Weighted Graphs

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Abstract

Given a large weighted graph $G = (V, E)$ and a subset U of V, we define several graphs with vertex set U in which two vertices are adjacent if they satisfy some prescribed proximity rule. These rules use the shortest path distance in G and generalize the proximity rules that generate some of the most common proximity graphs in Euclidean spaces. We prove basic properties of the defined graphs and provide algorithms for their computation.

1 Introduction

In Euclidean spaces, proximity graphs are a key tool to obtain neighborhood relations in a given set of points [5]. They have been intensively explored in the contexts of spacial distribution analysis [9] and graph drawing [7], among others.

In non-Euclidean settings, the Delaunay graph and its relatives have found applications in the analysis of networks that model real connection nets. A prominent example is the network Voronoi diagram (see Section 3.8 in [9]).

Here we deal with a complex graph G with a large number of vertices and edges, in which it is difficult to distinguish which are the relations of proximity among a subset of the vertices. The edges of the graph come with an associated positive weight. We study relations of proximity based on shortest paths along $G = (V, E)$ among the vertices of a subset $U \subseteq V$, which might represent the schools in the map of a city, the corresponding stations in a huge transportation net, etc. We use generalizations of some well-known proximity graphs. This appears as a natural method to provide notions of closeness.

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The natural and important question of defining suitable notions of closeness among vertices of a graph has found different kinds of answers in the literature. However, we are only aware of one approach that uses proximity graphs (see [6, 11]). The graphs considered there are clearly different from ours, as proximity is constructed by adopting a notion whose universe is a given geometric graph, but where the relations are given by the full Euclidean plane.

Let us mention that the set U together with the shortest-path distance constitutes a finite metric space, so some of the proximity graphs we consider are not new because they can be seen as a particular case of proximity graphs defined on general metric spaces. Even though there exists some literature on proximity graphs in metric spaces, to the best of our knowledge this topic has not been deeply investigated, as only some definitions and basic properties have been given (see Section 4.5 in [12], and also [4]). The sphere-of-influence graph has been further studied [3, 8], but it is out of the scope of our work.

When using empty regions as proximity criteria in G, such as disks, two main variations arise, since we might allow these disks to be centered at any point in G, or we might restrict their centers to lie only on vertices of the graph, as in [3, 1]. Moreover, the definition of certain regions of interference might depend on the multiplicity of paths or distances in G. Degeneracies that occur in the standard geometric case also generate several possibilities. For the sake of clarity we first present the situation where there are essentially no degeneracies (Sections 2–5). In Section 6 we drop the non-degeneracy assumptions and extend our results to the general setting.

Proofs and descriptions of the algorithms will be given in the full-version of this paper.

2 Definitions and Notation

We deal with a connected and edge-weighted graph $G = (V, U, E)$, where $U \subseteq V$ and all edges have positive real weights assigned to them. We assume that it is possible to consider points in the edges of G; more precisely, for every edge $e = (v_1, v_2)$ with weight $w(e)$ and every $r \in (0, w(e))$, we assume that there exists a point p in e and paths from both v_1 and v_2 to p such that the weight of the path from v_1 to p is r, and the weight of the path from v_2 to p is $w(e) - r$ (if G is embedded in the plane, these paths are simply portions of the edges). We say that p is a *point of* G if p is either a vertex of G, or a point in an edge of G. The *distance* $d_G(p,q)$ between two points p and q in G is defined as the minimum total weight of any path connecting p and q in G. The *closed disk* $D_G(p,r)$ is defined as the set of points q of G for which $d_G(p,q) \leq r$. We say that $u_i \in U$ is a *nearest neighbor* of $u_j \in U$ with $i \neq j$ if $d_G(u_j,u_i) \leq d_G(u_j,u_k)$ for all vertices $u_k \neq u_j \in U$. A *midpoint* of two points p and q of G is a point m on one of the shortest paths from p to q such that $d_G(m,p) = d_G(m,q)$. We denote the set of midpoints of p and q by $M_G(p,q)$. For the remainder of this paper, we define $|V| = m$, $|U| = n$, and $|E| = e$.

We first consider the case where the following non-degeneracy assumptions hold: (A1) for all $u_i, u_j \in U$, the shortest path connecting u_i and u_j is unique; (A2) there do not exist three distinct vertices $u_i, u_j \in U$, $v \in V - U$ such that $d_G(v,u_i) = d_G(v,u_j)$; (A3) there do not exist vertices $v_i, v_j \in V$, $u_i, u_j \in U$ such that $d_G(v_i,u_i) = d_G(v_j,u_j)$ with $v_i \neq u_i$; (A4) all paths in G between distinct nodes in V have different lengths.

Obviously, the previous assumptions are not independent, but considering them separately allows to clarify and provide a more precise description of the scenario. In Section 6, we extend the results from Sections 3–5 to the general case where A1–A4 are not necessarily satisfied.

We now adapt several known definitions to proximity structures in graphs $G = (V,U,E)$.

Definition 1 *The* nearest neighbor graph *of* $G = (V,U,E)$*, denoted by* $\mathrm{NNG}(G)$*, is the graph* $H = (U,F)$ *such that* $(u_i,u_j) \in F$ *if* u_j *is one of the nearest neighbors of* u_i *in* G*.*

Definition 2 *A* minimal spanning tree *of* $G = (V,U,E)$ *is a tree* $T = (U,F)$ *such that the sum of* $d_G(u_i,u_j)$ *over all edges* $(u_i,u_j) \in F$ *is minimal. The* union of the minimal spanning trees *of* G*, denoted by* $\mathrm{UMST}(G)$*, is the graph consisting of all the edges included in any of the minimal spanning trees of* G*.*

If A3 holds, each vertex in U has exactly one nearest neighbor and the minimal spanning tree of G, denoted by $\mathrm{MST}(G)$, is unique.

Definition 3 *The* relative neighborhood graph *of* $G = (V,U,E)$*, denoted by* $\mathrm{RNG}(G)$*, is the graph* $H = (U,F)$ *such that* $(u_i,u_j) \in F$ *if there exists no vertex* $u_k \in U$ *such that* $d_G(u_k,u_i) < d_G(u_i,u_j)$ *and* $d_G(u_k,u_j) < d_G(u_i,u_j)$*.*

Definition 4 *The* free Gabriel graph *of* $G = (V,U,E)$*, denoted by* $\mathrm{GG_f}(G)$*, is the graph* $H = (U,F)$ *such that* $(u_i,u_j) \in F$ *if there exists no vertex* $u_k \in U$ $(u_k \neq u_i,u_j)$ *such that* $d_G(p,u_k) \leq d_G(p,u_i)$*, where* p *is the midpoint of* u_i *and* u_j*.*

If A1 holds, there exists only one midpoint of u_i and u_j, thus the previous graph is well-defined.

Definition 5 *The* constrained Gabriel graph *of* $G = (V,U,E)$*, denoted by* $\mathrm{GG_c}(G)$*, is the graph* $H = (U,F)$ *such that* $(u_i,u_j) \in F$ *if the smallest closed disk centered at a vertex in* V *enclosing* u_i *and* u_j *does not contain any other vertex from* U*.*

The previous graph is well-defined if A3 holds.

Definition 6 *The* Voronoi region *of a vertex* $u_i \in U$ *is the set of points* p *of* G *such that* $d_G(p,u_i) \leq d_G(p,u_j)$ *for all vertices* $u_j \in U$ *different from* u_i*. The* Voronoi diagram *of* $G = (V,U,E)$*, denoted by* $\mathrm{VD}(G)$*, is the Voronoi diagram of the vertex set* U *for the distance* d_G*.*

Definition 7 *The* free Delaunay graph *of* $G = (V,U,E)$*, denoted by* $\mathrm{DG_f}(G)$*, is the graph* $H = (U,F)$ *such that* $(u_i,u_j) \in F$ *if there exists a closed disk* $D_G(p,r)$*, where* p *is a point of* G*, enclosing* u_i *and* u_j *and no other vertex from* U*.*

Definition 8 *The* constrained Delaunay graph *of* $G = (V,U,E)$*, denoted by* $\mathrm{DG_c}(G)$*, is the graph* $H = (U,F)$ *such that* $(u_i,u_j) \in F$ *if there exists a closed disk* $D_G(v,r)$*, with* $v \in V$*, enclosing* u_i *and* u_j *and no other vertex from* U*.*

3 Inclusion Sequence

The graphs just defined satisfy some inclusion relations. In this section we show which proximity graphs are subgraphs of which other proximity graphs assuming A1, A2, and A3.

Theorem 1 *The relations of containment among all classes of proximity graphs are shown in Table 1. The symbol* $\subseteq$ *means that the inclusion is satisfied for all graphs* G*, and* $\nsubseteq$ *means that there are graphs* G *for which the inclusion is not satisfied.*

All inclusions in the table are proper, in the sense that there exist graphs G for which the corresponding proximity subgraph does not coincide with its supergraph.

4 Geometric and Combinatorial Properties

We define the *dual graph of the Voronoi diagram* of $G = (V,U,E)$ as the graph with vertex set U and edges connecting two vertices if their Voronoi regions share some point in G that does not belong to the Voronoi region of any other element in U.

Table 1: Relations of containment among proximity graphs in the non-degenerate case.

	MST	RNG	GG_c	GG_f	DG_c	DG_f
NNG	$\subseteq$	$\subseteq$	$\nsubseteq$	$\subseteq$	$\subseteq$	$\subseteq$
MST		$\subseteq$	$\nsubseteq$	$\subseteq$	$\nsubseteq$	$\subseteq$
RNG			$\nsubseteq$	$\subseteq$	$\nsubseteq$	$\subseteq$
GG_c				$\nsubseteq$	$\subseteq$	$\subseteq$
GG_f					$\nsubseteq$	$\subseteq$
DG_c						$\subseteq$

Table 2: Running times of the algorithms to compute the proximity graphs on G.

proximity graph	running time
NNG	$O(e + (m-n)\log(m-n))$
MST	$O(e\,\alpha(e,n) + (m-n)\log(m-n))$
RNG	$O(\mathrm{APSP}(G) + \min\{n^2, e\}n)$
GG_c	$O(\mathrm{APSP}(G) + \min\{n^2, e\}m)$
GG_f	$O\left(\mathrm{APSP}(G) + \min\{n^2, e\}m\right)$
DG_c	$O(e + m\log m)$
DG_f	$O(e + (m-n)\log(m-n))$

Proposition 2 *Let $G = (V, U, E)$ be a graph. Then $\mathrm{DG_f}(G)$ is the dual the graph of $\mathrm{VD}(G)$.*

The previous proposition allows to draw the first analogy between the usual proximity graphs and these new proximity structures on graphs. Moreover, it is a key tool to prove the following result:

Corollary 3 *Let $G = (V, U, E)$ be a graph. The number of edges of $\mathrm{NNG}(G)$, $\mathrm{MST}(G)$, $\mathrm{RNG}(G)$, $\mathrm{GG_c}(G)$, $\mathrm{GG_f}(G)$, $\mathrm{DG_c}(G)$, and $\mathrm{DG_f}(G)$ is at most e.*

This bound is tight up to a constant factor:

Proposition 4 *There exists a graph $G = (V, U, E)$ such that $\mathrm{RNG}(G) = \mathrm{GG_f}(G) = \mathrm{DG_f}(G) = G$. There also exists a graph $G' = (V', U', E')$ such that the number of edges of $\mathrm{GG_c}(G')$ and $\mathrm{DG_c}(G')$ is $e'/2$. Furthermore, all of these graphs have $\Theta(n^2)$ edges.*

In the following theorems we show that the proximity graphs inherit planarity and acyclicity from the original graph.

Theorem 5 *Let $G = (V, U, E)$ be a planar graph. Then $\mathrm{NNG}(G)$, $\mathrm{MST}(G)$, $\mathrm{RNG}(G)$, $\mathrm{GG_c}(G)$, $\mathrm{GG_f}(G)$, $\mathrm{DG_c}(G)$, and $\mathrm{DG_f}(G)$ are planar.*

Theorem 6 *Let $G = (V, U, E)$ be a tree. Then $\mathrm{GG_c}(G)$ and $\mathrm{DG_c}(G)$ are forests, and $\mathrm{RNG}(G) = \mathrm{GG_f}(G) = \mathrm{DG_f}(G) = \mathrm{MST}(G)$.*

Next we give complete characterizations for those graphs that are isomorphic to a certain proximity graph of some other graph.

Proposition 7 *If $G = (V, E)$ is a graph, there exists a graph $\bar{G} = (\bar{V}, \bar{U}, \bar{E})$ such that $G \cong \mathrm{NNG}(\bar{G})$ if and only if G is acyclic and does not contain isolated vertices.*

Proposition 8 *If $G = (V, E)$ is a graph, there exists a graph $\bar{G} = (\bar{V}, \bar{U}, \bar{E})$ such that $G \cong \mathrm{MST}(\bar{G})$ if and only if G is a tree.*

Proposition 9 *If $G = (V, E)$ is a graph, there exists a graph $\bar{G} = (\bar{V}, \bar{U}, \bar{E})$ such that $G \cong \mathrm{RNG}(\bar{G})$ if and only if G is triangle-free.*

Proposition 10 *Let $G = (V, E)$ be a graph. There exists a graph $\bar{G} = (\bar{V}, \bar{U}, \bar{E})$ such that $G \cong \mathrm{GG_c}(\bar{G}) = \mathrm{GG_f}(\bar{G}) = \mathrm{DG_c}(\bar{G}) = \mathrm{DG_f}(\bar{G})$.*

5 Algorithms

We have derived algorithms to compute each of the proximity graphs we have studied. Due to lack of space, we omit the description of the algorithms and only give their running times.

In some cases the algorithm computes the shortest paths between all pairs of vertices in U. If G is a sparse graph, we use the algorithm in [10], which runs in $O(m\log m + ne\log\alpha(m,e))$ time. If G is dense, we use the algorithm in [2], which runs in $O\left(m^3\log^3\log m/\log^2 m\right)$ time. We define $\mathrm{APSP}(G) = \min\{m\log m + ne\log\alpha(m,e), m^3\log^3\log m/\log^2 m\}$.

Theorem 11 *For each graph $G = (V, U, E)$, the proximity graphs on G can be computed in the number of steps indicated in Table 2.*

6 Presence of Degeneracies

In this section we generalize our results to the case in which degeneracies arise.

First of all, we look through the definitions. The graphs $\mathrm{NNG}(G)$, $\mathrm{UMST}(G)$, $\mathrm{RNG}(G)$, $\mathrm{DG_f}(G)$, and $\mathrm{DG_c}(G)$ are well-defined regardless of the properties of G, although, in contrast to the non-degenerate case, a vertex in U might have several nearest neighbors.

In the general case there might be more than one shortest path between two vertices of U. This gives rise to two definitions of free Gabriel graphs:

Definition 9 *The* free-one Gabriel graph *of $G = (V, U, E)$, denoted by $\mathrm{GG_{f1}}(G)$, is the graph $H = (U, F)$ such that $(u_i, u_j) \in F$ if there exists $p \in$*

Table 3: Relations of containment among all classes of proximity graphs in the general case.

	UMST	RNG	$\mathrm{GG_{ca}}$	$\mathrm{GG_{c1}}$	$\mathrm{GG_{fa}}$	$\mathrm{GG_{f1}}$	$\mathrm{DG_{c}}$	$\mathrm{DG_{f}}$
NNG	$\subseteq$	$\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$
UMST		$\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$
RNG			$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$
$\mathrm{GG_{ca}}$				$\subseteq$	$\not\subseteq$	$\not\subseteq$	$\subseteq$	$\subseteq$
$\mathrm{GG_{c1}}$					$\not\subseteq$	$\not\subseteq$	$\subseteq$	$\subseteq$
$\mathrm{GG_{fa}}$						$\subseteq$	$\not\subseteq$	$\subseteq$
$\mathrm{GG_{f1}}$							$\not\subseteq$	$\subseteq$
$\mathrm{DG_{c}}$								$\subseteq$

$M_G(u_i, u_j)$ such that no vertex $u_k \in U$ ($u_k \neq u_i, u_j$) satisfies $d_G(p, u_k) \leq d_G(p, u_i)$.

Definition 10 *The* free-all Gabriel graph *of $G = (V, U, E)$, denoted by $\mathrm{GG_{fa}}(G)$, is the graph $H = (U, F)$ such that $(u_i, u_j) \in F$ if, for each $p \in M_G(u_i, u_j)$, no vertex $u_k \in U$ ($u_k \neq u_i, u_j$) satisfies $d_G(p, u_k) \leq d_G(p, u_i)$.*

Analogously, the definition of the constrained Gabriel graph must be replaced by the following variants:

Definition 11 *The* constrained-one Gabriel graph *of $G = (V, U, E)$, denoted by $\mathrm{GG_{c1}}(G)$, is the graph $H = (U, F)$ such that $(u_i, u_j) \in F$ if there exists a closed disk $D_G(v, r)$, with $v \in V$ and $r = \min_{v \in V}\{r \mid D_G(v, r)$ contains both u_i and $u_j\}$, enclosing u_i and u_j and no other vertex from U.*

Definition 12 *The* constrained-all Gabriel graph *of $G = (V, U, E)$, denoted by $\mathrm{GG_{ca}}(G)$, is the graph $H = (U, F)$ such that $(u_i, u_j) \in F$ if every closed disk $D_G(v, r)$ containing both u_i and u_j, and where $v \in V$ and $r = \min_{v \in V}\{r \mid D_G(v, r)$ contains both u_i and $u_j\}$, does not contain any other vertex of U.*

Now we may go through the inclusion relations of the proximity graphs.

Theorem 12 *If degenerate situations are allowed, the relations of containment among all classes of proximity graphs are shown in Table 3. Furthermore, all classes of proximity graphs are different.*

To conclude this section, we focus on the most important properties presented in Section 3.

The fact that $\mathrm{DG_f}(G)$ is the dual graph of the Voronoi diagram of G holds in all cases. On the other hand, if A2 is not satisfied, some of the proximity graphs might have more edges than the original graph:

Theorem 13 *Let $G = (V, U, E)$ be a graph. The number of edges of $\mathrm{GG_{ca}}(G)$, $\mathrm{GG_{c}}(G)$, $\mathrm{GG_{fa}}(G)$, $\mathrm{GG_{f}}(G)$, $\mathrm{DG_{c}}(G)$, and $\mathrm{DG_{f}}(G)$ is at most e. The number of edges of $\mathrm{NNG}(G)$, $\mathrm{UMST}(G)$, and $\mathrm{RNG}(G)$ may be greater than e.*

Finally, we check whether all proximity graphs inherit the property of being planar or acyclic in the degenerate case.

Theorem 14 *Let $G = (V, U, E)$ be a planar graph. Then the graphs $\mathrm{GG_{ca}}(G)$, $\mathrm{GG_{c1}}(G)$, $\mathrm{GG_{fa}}(G)$, $\mathrm{GG_{f1}}(G)$, $\mathrm{DG_{c}}(G)$, and $\mathrm{DG_{f}}(G)$ are planar, whereas $\mathrm{NNG}(G)$, $\mathrm{UMST}(G)$, and $\mathrm{RNG}(G)$ may not be.*

Theorem 15 *Let $G = (V, U, E)$ be a tree. Then the graphs $\mathrm{GG_{ca}}(G)$, $\mathrm{GG_{c1}}(G)$, $\mathrm{GG_{fa}}(G)$, $\mathrm{GG_{f1}}(G)$, $\mathrm{DG_{c}}(G)$, and $\mathrm{DG_{f}}(G)$ are acyclic, whereas $\mathrm{NNG}(G)$, $\mathrm{UMST}(G)$, and $\mathrm{RNG}(G)$ may not be.*

The algorithms in the preceding section can be adapted to run under the presence of degeneracies yet we omit here further details.

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