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Recent developments on the number of $(\leq k)$-sets, halving lines, and the rectilinear crossing number of K_n.

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Resumen

We present the latest developments on the number of $(\leq k)$-sets and halving lines for (generalized) configurations of points; as well as the rectilinear and pseudolinear crossing numbers of K_n. In particular, we define *perfect* generalized configurations on n points as those whose number of $(\leq k)$-sets is exactly $3\binom{k+1}{2}$ for all $k \leq n/3$. We conjecture that for each n there is a perfect configuration attaining the maximum number of $(\leq k)$-sets and the pseudolinear crossing number of K_n. We prove that for any $k \leq n/2$ the number of $(\leq k)$-sets is at least $3\binom{k+1}{2} + 3\binom{k-\lfloor n/3 \rfloor+1}{2} + 18\binom{k-\lceil 4n/9 \rceil+1}{2} - O(n)$. This in turn implies that the pseudolinear (and consequently the rectilinear) crossing number of any perfect generalized configuration on n points is at least $\frac{277}{729}\binom{n}{4} + O(n^3) \geq 0.379972\binom{n}{4} + O(n^3)$.

1 Introduction

Let P be a set of n points in general position in the plane. A subset of P consisting of $k \leq n/2$ points is called a *k-set* if it can be separated by the rest of P by a straight line. Any j-set with $j \leq k$ is called a *$\leq k$-set*. We denote by $\chi_k(P)$ and $\chi_{\leq k}(P)$ the number of k-sets and $\leq k$-sets of P, respectively. The number of edge crossings in the drawing of the complete graph K_n whose set of vertices is P and whose edges are straight line segments is denoted by $\overline{cr}(P)$. This is called the *rectilinear crossing number* of P. An edge in such a graph is called a *k-edge* if it leaves exactly k points of P on one side. When n is even the $(n/2-1)$-edges are known as *halving lines*, since they divide the remaining $n-2$ points of P in half. When n is odd the $(n-3)/2$-edges are also called halving lines since they divide P almost in half. As before, any j-edge with $j \leq k$ is called a *$\leq k$-edge*. Let $\eta_k(P)$ and $\eta_{\leq k}(P)$ be the number of k-edges and $\leq k$-edges of P, respectively, and $h(P) = \eta_{\lfloor n/2 \rfloor - 1}$ the number of halving lines of P.

The problems of finding the minimum number of $\leq k$-sets or $\leq k$-edges, the maximum number of halving lines, and the minimum crossing number of P over all configurations P of n points in the plane have been widely studied [8]. In other words, we want to estimate the values of

$$\chi_{\leq k}(n) = \min_{|P|=n} \chi_{\leq k}(P), \eta_{\leq k}(n) = \min_{|P|=n} \eta_{\leq k}(P), h(n) = \max_{|P|=n} h(P), \overline{cr}(n) = \min_{|P|=n} cr(P)$$

where the minima and maximum are taken over all sets P of n points in the plane. The last function $\overline{cr}(n)$ is known as the *rectilinear crossing number* of K_n.

All these problems are closely related. Note that there is a one-to-one correspondence between the set of k-sets and the set of $(k-1)$-edges of P, i.e., $\chi_k(P) = \eta_{k-1}(P)$, and thus $\chi_{\leq k}(n) = \eta_{\leq k-1}(n)$. Since all $\binom{n}{2}$ edges associated with P are either $(\leq \lfloor n/2 \rfloor - 2)$-edges or halving lines then

$$h(n) = \binom{n}{2} - \eta_{\leq \lfloor n/2 \rfloor - 2}(n) = \binom{n}{2} - \chi_{\leq \lfloor n/2 \rfloor - 1}(n).$$

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Ábrego and Fernández-Merchant [5] and independently Lovász et al. [11], proved the following relationship between the crossing number and the number of k-edges:

$$cr(P) = 3\binom{n}{4} - \sum_{k=1}^{\lfloor n/2 \rfloor} (k-1)(n-k-1)\chi_k(P), \text{ or equivalently}$$

$$cr(P) = \sum_{k=1}^{\lfloor n/2 \rfloor - 1} (n-2k-1)\chi_{\le k}(P) - \frac{3}{4}\binom{n}{3} + \left(1 + (-1)^{n+1}\right)\frac{1}{8}\binom{n}{2}. \tag{1}$$

All these concepts and results can be extended to *generalized configurations of points.* A set P of n points in the plane can be encoded by a *circular sequence* Π (see below) as follows: Label the points of P from 1 to n. Draw a circle containing P together with a directed tangent line l. Project P onto l to obtain an ordering of P, this corresponds to a permutation of the elements of $\{1, 2, 3, ..., n\}$. Rotate l around the circle (in both directions) and record all permutations. As a result we obtain a doubly-infinite sequence of permutations of the elements of $\{1, 2, ..., n\}$ with period $2\binom{n}{2}$.

In general, a *circular sequence* is a doubly infinite sequence $(..., \pi_{-1}, \pi_0, \pi_1, ...)$ of permutations on n elements, such that any two consecutive permutations π_i and π_{i+1} differ by a transposition τ_i of neighboring elements, and such that for every j, π_j is the reversed permutation of $\pi_{j+\binom{n}{2}}$. Circular sequences were introduced by Goodman and Pollack [10], [9] who established a one-to-one correspondence between circular sequences and generalized configurations of points, that is, configurations of $\binom{n}{2}$ pseudolines and n points where each pseudoline passes through exactly two points and two pseudolines intersect exactly once. When all the pseudolines can be straight lines the generalized configuration is called *stretchable* and it corresponds to a configuration of points in the plane. Thus every configuration of points in the plane corresponds to a circular sequence but only stretchable circular sequences correspond to sets of points in the plane. Any subsequence of Π consisting of $\binom{n}{2}$ consecutive permutations is called a *halfperiod.* If τ_j occurs between elements in positions i and $i+1$ we say that τ_j is an *i-transposition.* If $i \le n/2$ then any i-transposition or $(n-i)$-transposition is called *i-critical.* The k-sets of Π are precisely the subsets of $\{1, 2, ..., n\}$ of size k that occupy the first or last k positions in a permutation of $\{1, 2, ..., n\}$. (These k-sets coincide with those defined for configurations of points when Π is stretchable.) The set of k-sets of Π is then determined by the set of k-critical transpositions in a halfperiod of Π. In fact a k-critical transposition is a $(k-1)$-pseudoedge. Thus $\chi_k(\Pi)$ and $\chi_{\le k}(\Pi)$ are the number of k-critical, and respectively $(\le k)$-critical, transpositions in any halfperiod of Π and (1) still holds. So now we can define $\chi_{\le k}(n), \eta_{\le k}(n), \widetilde{h}(n)$, and $\widetilde{cr}(n)$ by optimizing over all *generalized* configurations of n points.

2 Summary of recent results

By the end of 2006 the exact values of $h(n), \widetilde{h}(n), \overline{cr}(n)$, and $\widetilde{cr}(n)$ were only known for $n \le 19$ and $n = 21$, except for $\widetilde{h}(14)$ and $\widetilde{h}(16)$. We have managed to obtain the exact values for $n \le 27$.

n	14	16	18	20	22	23	24	25	26	27
$h(n) = \widetilde{h}(n)$	22*	27	33	38	44	75	51	85	57	96
$\overline{cr}(n) = \widetilde{cr}(n)$	324*	603*	1029*	1657	2528	3077	3699	4430	5250	6180

* Previously known values for the geometric case.

This improvement was an application of the following theorem that concentrates on the central behavior of circular sequences:

Theorem 2.1. *Let Π be a circular sequence associated to a generalized configuration of n points. Then*

$$\chi_{\lfloor n/2 \rfloor}(\Pi) \le \begin{cases} \left\lfloor \frac{1}{2}\binom{n}{2} - \frac{1}{2}\chi_{\le \lfloor n/2 \rfloor - 2}(\Pi) \right\rfloor & \textit{if } n \textit{ is even,} \\ \left\lfloor \frac{2}{3}\binom{n}{2} - \frac{2}{3}\chi_{\le \lfloor n/2 \rfloor - 2}(\Pi) + \frac{1}{3} \right\rfloor & \textit{if } n \textit{ is odd.} \end{cases}$$

In terms of general bounds, Ábrego and Fernández-Merchant [4] proved the following upper bound for $\overline{cr}(n)$, and therefore for $\widetilde{cr}(n)$. Let P be a set of N points in the plane and H its set of halving lines. Consider the bipartite graph $G = (P, H)$ where $p \in P$ is adjacent to $l \in H$ if p is on l. A matching of G saturating P is called a *halving-line matching* of P.

Theorem 2.2. *If P is a N-element point set in general position, with N even, and P has a halving-line matching; then*

$$\widetilde{cr}(n) \leq \overline{cr}(n) \leq \left(\frac{24\,\mathrm{cr}(P) + 3N^3 - 7N^2 + (30/7)N}{N^4}\right)\binom{n}{4} + \Theta(n^3).$$

The best upper bound based on this result was obtained using the best known construction for $N = 90$ [6],

$$\overline{cr}(n) \leq 0.380548\binom{n}{4} + \Theta(n^3).$$

On the other hand, Ábrego and Fernández-Merchant [5] and independently Lovász et al. [11], improved the previously known lower bound for $\chi_{\leq k}(n)$ to

$$\chi_{\leq k}(n) \geq 3\binom{k+1}{2}. \tag{2}$$

This bound was improved, by Aichholzer et al. [7] in the rectilinear case and generalized to the pseudolinear case by Ábrego et al. [2], to

$$\chi_{\leq k}(n) \geq 3\binom{k+1}{2} + 3\binom{k - \lfloor n/3 \rfloor + 1}{2} + O(n). \tag{3}$$

As a consequence, using (1), the best known lower bound for the rectilinear and pseudolinear crossing numbers satisfies

$$\overline{cr}(n) \geq \widetilde{cr}(n) \geq 0.37968\binom{n}{4} + O\left(n^3\right).$$

It is known that (2) is tight for $k \leq n/3$ and moreover, Ábrego et al. [3] proved the following

Theorem 2.3. *If a generalized configuration of n points Π attains $\widetilde{cr}(n)$ and $\chi_{\leq \lfloor n/3 \rfloor}(\Pi) = 3\binom{\lfloor n/3 \rfloor + 1}{2}$ then $\chi_{\leq k}(\Pi) = 3\binom{k+1}{2}$ for all $k \leq n/3$.*

A configuration Π that satisfies $\chi_{\leq k}(\Pi) = 3\binom{k+1}{2}$ for all $k \leq n/3$ is called *perfect*. We say that a configuration of n points achieving $\widetilde{cr}(n)$ is *crossing optimal*. We believe that

Conjecture 2.4. If Π is crossing optimal then it is perfect.

The following weaker version of this conjecture would still lead to general lower bound improvements using Theorem 2.6.

Conjecture 2.5. For any n there is a crossing optimal configuration that is perfect.

Here we improve the lower bound for $\chi_{\leq k}(\Pi)$ and therefore for the pseudolinear crossing number for perfect configurations.

Theorem 2.6. *If Π is a perfect generalized configuration of n points then for all $k \leq n/2$,*

$$\chi_{\leq k}(\Pi) \geq 3\binom{k+1}{2} + 3\binom{k - \lfloor n/3 \rfloor + 1}{2} + 18\binom{k - \lfloor 4n/9 \rfloor + 1}{2} + O(n) \tag{4}$$

In fact we prove a stronger result. A point that belongs to a k-set but not to a $\leq (k-1)$-set is said to be in the k^{th} *layer* of Π. Let L_k denote the k^{th}-layer of Π . We say that Π is 3-*regular* if there are exactly 3 points in L_k for all $k \leq n/3$.

Theorem 2.7. *If* Π *is perfect then* Π *is* 3*-regular.*

Theorem 2.8. *If* Π *is a* 3*-regular generalized configuration of* n *points and* $18 \mid n$ *then*

$$\chi_{\leq k}(\Pi) \geq 3\binom{k+1}{2} + 3\binom{k-n/3+1}{2} + 18\binom{k-4n/9+1}{2} + \begin{cases} 3 & \text{if } k \geq 4n/9 \\ 0 & \text{else.} \end{cases} \tag{5}$$

The previous two theorems imply (4). Also (4)and (1) imply that the pseudolinear, and consequently the rectilinear crossing number of any *perfect* configuration on n points is $\geq \frac{277}{729}\binom{n}{4} + O\left(n^3\right) \geq 0.379972\binom{n}{4} + O\left(n^3\right)$.

3 Proof of Theorem 2.8

For each $1 \leq p \leq n$ let $L(p)$ be the smallest *position* of p in a permutation of Π. Then for $k \leq n/2$, $L_k = \{p \in P : L(p) = k\}$. Note that P is the disjoint union of its *layers* (some may be empty). Let $l_i = |L_i|$ and consider the partial sums $s_k = l_1 + l_2 + ... + l_k$. Then $n \geq s_k \geq 2k+1$ for all $1 \leq k \leq n/2$ since the first and last k elements in any term of Π belong to $L_1 \cup ... \cup L_k$ and at least one more element must enter this region. In particular $s_1 = l_1 \geq 3$ and $s_{\lfloor n/2 \rfloor} = n$.

For each point $p \in P$ we follow the transpositions of p in a fixed halfperiod. The transposition $\{p, q\}$ may have a different role when following p that when following q. Thus we use ordered pairs. We say that (q, p) is a *transposition of* p.

Let $p \in P$ and fix a halfperiod $\pi(p)$ satisfying that if $p \in L_i$ then the first row of $\pi(p)$ shows p in the i^{th} position. This naturally orders the $n-1$ transpositions of p according to the order in which they occur in $\pi(p)$. Following this order, we say that a transposition of p is a *forth-transposition* if p moves to a larger position (from left to right) in $\pi(p)$ and a *back-transposition* otherwise. The first j-forth-transposition of p is called j*-primary.* A pair formed by a j-back-transposition of p and the next j-transposition of p (which must be a nonprimary forth-transposition) is called a j*-secondary pair* of p. Then for $j \leq n/2$ we can say that a j- or $(n-j)$-secondary pair is a j-critical pair.

For $p_1 \in P$, we write $(p_0, p_1) \to (p_1, p_2)$ if $\{(p_0, p_1), (p_2, p_1)\}$ is a secondary pair of p_1 with back-transposition (p_2, p_1). If $p_1 \in L_i$ then p_1 moves from position i to position $n+1-i$ in $\pi(p_1)$. Thus there is exactly one j-primary transposition of p_1 for all $i \leq j \leq n-i$. Moreover, (p, p_1) is a back-transposition only if the first row of $\pi(p_1)$ shows p in one of the first $i-1$ positions. This means that there are exactly $i-1$ secondary pairs of p_1 and if $(p_0, p_1) \to (p_1, p_2)$ with $p_2 \in L_j$ then $j < i$. Thus (p_1, p_2) must be a forth-transposition of p_2. If $p_1 \in L_i, p_r \in L_j$, and

$$(p_0, p_1) \to (p_1, p_2) \to (p_2, p_3) \to ... \to (p_{r-1}, p_r) \tag{6}$$

then we say that (p_0, p_1) *goes from* L_i *to* L_j *in* r *steps.* Note that if r is as large as possible then (p_{r-1}, p_r) is a k-primary transposition of p_r for some $1 \leq k \leq n/2$ and all the transpositions in (6) are k-critical. In this case we say that the forth-transposition (p_0, p_1) has *rank* r and write $rank(p_0, p_1) = r$. Then all primary transpositions have rank 1. The *rank of a secondary pair* is the rank of its forth-transposition. Let

$$\chi_{\leq k}(\Pi, r) = \#(\leq k)\text{-critical rank } r \text{ transpositions of } \Pi.$$

Then $\chi_{\leq k}(\Pi, 1) = \#(\leq k)$-critical primary transposition and since each forth-transposition of rank ≥ 2 belongs to a secondary pair then $\chi_{\leq k}(\Pi)$ can be expressed in terms of its forth-transpositions.

$$2\chi_{\leq k}(\Pi) = \chi_{\leq k}(\Pi, 1) + 2\sum_{r=2}^{\lfloor n/2 \rfloor} \chi_{\leq k}(\Pi, r). \tag{7}$$

Based on the fact that all transpositions in (6) occur in the same position, we keep track of the forth-transpositions using the following notation. For $1 \leq j \leq i \leq n/2$ and $1 \leq r \leq i-j+1$ let

$F_r(i,j)$ be the set of forth-transpositions that go from L_i to L_j in r steps, and $M_r(i,j)$ the set of those elements in $F_r(i,j)$ with rank r. If I is a set of indices then

$$F_r(I,j) = \bigcup_{i\in I} F_r(i,j) \text{ and } M_r(I,j) = \bigcup_{i\in I} M_r(i,j).$$

Let $I_j = \{j, j+1, j+2, ..., \lfloor n/2 \rfloor\}$.

Lemma 3.1. *For all* $1 \le r \le n/2$

$$\chi_k(\Pi, r) \ge \sum_{j=1}^{\lfloor n/2 \rfloor - 1} \max\left(|M_r(I_j,j)| - l_j(n-1-2k), 0\right).$$

Proof. By definition, if $(p_0,p_1) \in M_r(I_j,j)$ and $(p_0,p_1) \to (p_1,p_2) \to ... \to (p_{r-1},p_r)$ then (p_{r-1},p_r) is a primary transposition of L_j. This means that the number of h-critical transpositions in $M_r(I_j,j)$ is bounded above by the number of h-critical primary transpositions of L_j. Now, for each $p \in L_j$ and $j \le h \le n/2$ we have exactly one h-primary and one $(n-h)$-primary transposition of p, both of them are h-critical. Then there are l_j transpositions of L_j that are h-primary and l_j that are $(n-h)$-primary. Thus at most $l_j(n-1-2k)$ elements of $M_r(I_j,j)$ are not $(\le k)$-critical. □

Proof. (Theorem 2.8) Since Π is 3-regular then $l_j = 3$ and $s_j = 3j$ for all $1 \le j \le n/3$. If $j > n/3$ then $l_j = 0$ and $s_j = n$.

If $k < 4n/9$ then (5) coincides with (3). For $k \ge 4n/9$ we bound $\chi_{\le k}(\Pi,1) + \chi_{\le k}(\Pi,2) + \chi_{\le k}(\Pi,3)$ below. The number of k-critical primary transpositions of Π is $2(l_1 + l_2 + ... + l_k) = 2s_k$ then

$$\chi_{\le k}(\Pi,1) \ge 2\sum_{j=1}^{k} s_j = \sum_{j=1}^{n/3} 3j + \sum_{j=n/3+1}^{k} n = 3\binom{n/3+1}{2} + n(k - n/3). \tag{8}$$

By Lemma 3.1 applied to $r=2$ and $r=3$ (disregard the maximum and note that $2k - 8n/9 + 1 \le 2k - 7n/9 + 1 \le n/2 - 1$)

$$\chi_{\le k}(\Pi,2) + \chi_{\le k}(\Pi,3) \ge \sum_{j=1}^{2k-7n/9+1} |M_2(I_j,j)| + \sum_{j=1}^{2k-8n/9+1} |M_3(I_j,j)| - 3(n-1-2k)(4k - 5n/3 + 2). \tag{9}$$

Since there are exactly $3(j-1)$ secondary pairs of L_j, at most $3(j-1)$ transpositions in $F_3(I_j,j)$ continue to another layer after passing through L_j. This means

$$\sum_{j=1}^{2k-8n/9+1} |M_3(I_j,j)| \ge \sum_{j=1}^{2k-8n/9+1} \left(|F_3(I_j,j)| - 3(j-1)\right) = \sum_{j=1}^{2k-8n/9+1} |F_3(I_j,j)| - \sum_{j=1}^{2k-8n/9} 3j \tag{10}$$

The transpositions that go to L_i in 3 steps, $F_3(I_i,i)$, can be partitioned into the sets $F_2(I_j,j) \cap F_3(I_j,i)$ with $i+1 \le j \le n/2$ and thus

$$\begin{aligned}
\sum_{i=1}^{2k-8n/9+1} |F_3(I_i,i)| &= \sum_{i=1}^{2k-8n/9+1} \sum_{j=i+1}^{n/2} |F_2(I_j,j) \cap F_3(I_j,i)| \\
&= \sum_{j=1}^{2k-7n/9+1} |M_2(I_j,j)| + \sum_{j=2}^{2k-8n/9+2} \sum_{i=1}^{j-1} |F_2(I_j,j) \cap F_3(I_j,i)| \\
&+ \sum_{j=2k-8n/9+3}^{n/2} \sum_{i=1}^{2k-8n/9+1} |F_2(I_j,j) \cap F_3(I_j,i)|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{2k-7n/9+1}\left|M_2\left(I_j,j\right)\right| + \sum_{i=1}^{2k-8n/9+1}\left|F_3\left(I_i,i\right)\right| \\
&\geq \left|M_2\left(I_2,1\right)\right| + \sum_{j=2}^{2k-8n/9+2}\left(\left|M_2\left(I_j,j\right)\right| + \sum_{i=1}^{j-1}\left|F_2\left(I_j,j\right)\cap F_3\left(I_j,i\right)\right|\right) \\
&+ \sum_{j=2k-8n/9+3}^{2k-7n/9+1}\left(\left|M_2\left(I_j,j\right)\right| + \sum_{i=1}^{2k-8n/9+1}\left|F_2\left(I_j,j\right)\cap F_3\left(I_j,i\right)\right|\right) \qquad (11)
\end{aligned}$$

For fixed j note that $\bigcup_{i=1}^{j-1} F_2\left(I_j,j\right)\cap F_3\left(I_j,i\right)$ consists of those transpositions of rank ≥ 3 that first go to L_j and then continue to some L_i with $1\leq i\leq j-1$. Then

$$\left|M_2\left(I_j,j\right)\right| + \sum_{i=1}^{j-1}\left|F_2\left(I_j,j\right)\cup F_3\left(I_j,i\right)\right| = \left|F_2\left(I_j,j\right)\right|. \qquad (12)$$

If $h\leq j-2$ there are at most $3\left(j-1-h\right)$ transpositions that first go to L_j and then to one of the $j-1-h$ layers $L_{h+1},L_{h+2},...,L_{j-2},L_{j-1}$ and all these transpositions are in $F_2\left(I_j,j\right)$. Then

$$\left|M_2\left(I_j,j\right)\right| + \sum_{i=1}^{h}\left|F_2\left(I_j,j\right)\cap F_3\left(I_j,i\right)\right| \geq \left|F_2\left(I_j,j\right)\right| - 3\left(j-1-h\right).$$

In particular, for $j\geq 2k-8n/9+3$ and $h=2k-8n/9+1$ we have

$$\left|M_2\left(I_j,j\right)\right| + \sum_{i=1}^{2k-8n/9+1}\left|F_2\left(I_j,j\right)\cap F_3\left(I_j,i\right)\right| \geq \left|F_2\left(I_j,j\right)\right| + 6\left(k-4n/9+1\right) - 3j. \qquad (13)$$

Using (12), (13), and $M_2\left(I_2,1\right)=F_2\left(I_2,1\right)$ we bound (11) below

$$\begin{aligned}
&\sum_{j=1}^{2k-8n/9+2}\left|F_2\left(I_j,j\right)\right| + \sum_{j=2k-8n/9+3}^{2k-7n/9+1}\left(\left|F_2\left(I_j,j\right)\right| + 6\left(k-4n/9+1\right) - 3j\right) \qquad (14) \\
&= \sum_{j=1}^{2k-7n/9+1}\left|F_2\left(I_j,j\right)\right| + \sum_{j=2k-8n/9+3}^{2k-7n/9+1}\left(6\left(k-4n/9+1\right) - 3j\right).
\end{aligned}$$

Each point $p\in L_{h+1}\cup L_{h+2}\cup ...\cup L_{\lfloor n/2\rfloor}$ has at least h back-transpositions (q,p) with $q\in L_1\cup L_2\cup ...\cup L_h$ and each point $p\in L_j, 1\leq j\leq h$, has at least $j-1$ back-transpositions (q,p) with $q\in L_1\cup L_2\cup ...\cup L_h$. Thus

$$\sum_{j=1}^{h}\left|F_2\left(I_j,j\right)\right| \geq h\left(l_{h+1}+...+l_{\lfloor n/2\rfloor}\right) + \sum_{j=1}^{h}\left(j-1\right)l_j = \sum_{j=1}^{h}\left(l_{j+1}+...+l_{\lfloor n/2\rfloor}\right) = \sum_{j=1}^{h}\left(n-s_j\right). \quad (15)$$

Finally, (7), (8), (9), (10), (11), (14), and (15) imply

$$\begin{aligned}
\chi_{\leq k}\left(\Pi\right) &\geq \chi_{\leq k}\left(\Pi,1\right) + \chi_{\leq k}\left(\Pi,2\right) + \chi_{\leq k}\left(\Pi,3\right) \geq 3\binom{n/3+1}{2} + n\left(k-n/3\right) + \sum_{j=1}^{2k-7n/9+1}\left(n-3j\right) \\
&+ \sum_{j=2k-8n/9+3}^{2k-7n/9+1}\left(6\left(k-4n/9+1\right)-3j\right) - \sum_{j=1}^{2k-8n/9} 3j - 3\left(n-1-2k\right)\left(4k-5n/3+2\right) \\
&= 3\binom{k+1}{2} + 3\binom{k-n/3+1}{2} + 18\binom{k-4n/9+1}{2} + 3.
\end{aligned}$$

□

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