On Crossing Numbers of Geometric Proximity Graphs

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Abstract

Let P be a set of n points in the plane. A geometric proximity graph on P is a graph where two points are connected by a straight-line segment if they satisfy some prescribed proximity rule. We consider four classes of higher order proximity graphs, namely, the k-nearest neighbor graph, the k-relative neighborhood graph, the k-Gabriel graph and the k-Delaunay graph. For $k = 0$ ($k = 1$ in the case of the k-nearest neighbor graph) these graphs are plane, but for higher values of k in general they contain crossings. In this paper we provide lower and upper bounds on their minimum and maximum number of crossings. We give general bounds and we also study particular cases that are especially interesting from the viewpoint of applications. These cases include the 1-Delaunay graph and the k-nearest neighbor graph for small values of k.

Keywords Proximity graphs; geometric graphs; crossing number.

1 Introduction and basic notation

A *geometric graph on a point set* P is a pair $G = (P, E)$ in which the *vertex set* P is assumed to be in general position, i.e., no three points are collinear, and the set E of *edges* consists of straight-line segments with endpoints in P. Notice that the focus is more on the drawing rather than on the underlying graph, as carefully pointed out by Brass, Moser and Pach in their survey book ([6], page 373).

A *proximity graph* is a graph $G = (V, E)$ in which the nodes represent geometric objects in a given set, typically points, and two nodes are adjacent when the corresponding objects are considered to be neighbors according to some specific proximity criterion. A *geometric proximity graph* is a geometric graph in which the adjacency is decided by some neighborhood rule; they are also sometimes called *proximity drawings* [18]. Examples of these graphs are the *k-nearest neighbor graph*, k-$\mathsf{NNG}(P)$, in which every point is joined with a directed segment to its k closest neighbors, and the k-Delaunay graph, k-$\mathsf{DG}(P)$, in which p_i and p_j are connected with a segment if there is some circle through p_i and p_j that contains at most k points from P in its interior. Other similar definitions are given later in this paper.

Proximity graphs have been widely used in applications in which extracting shape or structure from a point set is a required tool or even the main goal, as is the case of computer vision, pattern

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recognition, visual perception, geographic information systems, instance-based learning, and data mining [13, 19, 26]. In the area of *graph drawing* [5, 12, 14] the main goal is to realize —or *to draw*— a given combinatorial graph as a geometric proximity graph, which leads to problems on characterizing the graphs that admit such a representation and designing efficient algorithms to construct the drawing whenever possible (see the survey [18] in this respect).

Usually graphs are drawn in the plane with points as nodes and Jordan arcs as edges. When two edges share an interior point, we say that there is a *crossing*. Both as a natural aesthetic measure for graph drawing and as a fundamental issue in the mathematical context, the number of crossings is a parameter that has been attracting extensive study. Given a graph G, the *crossing number of* G, denoted by $cr(G)$, is the minimum number of edge crossings in any drawing of G; if this number is 0, we say that the graph is *planar*. The *rectilinear crossing number of* G, denoted by $\overline{cr}(G)$, is the smallest number of crossings in any drawing of G in which the edges are represented by straight-line segments.

Computing the crossing number of a graph is an NP-hard problem [9], and both the generic and rectilinear crossing numbers of very fundamental graphs, such as the complete graph K_n and the complete bipartite graph $K_{m,n}$ are still unknown [29, 11]. These problems have been attracting a great amount of attention and recently a continuous chain of improvements has led progressively to narrow the gap between the lower and upper bounds [15, 3, 2]. There are also several results on the numbers of crossings that are sensitive to the size of the graph —particulary the *crossing lemma* [4, 17, 6]—, or to the exclusion of some configurations [6, 20, 22, 28, 8].

In this paper we study the crossing numbers of several higher order geometric proximity graphs related to Delaunay graphs. If P is a set of points in the plane, each of the proximity graphs we consider is a geometric graph on P that has some number of crossings that will be denoted by $\boxtimes(\ )$, and we investigate how this number varies when all possible point sets P in general position, with $|P| = n$, are considered. The generic conclusion that may be derived from our research is that this family of graphs has a relatively small number of crossings.

The fact that this specific issue has not been investigated previously is somehow surprising. As an explanation, one may first consider that 0-order proximity graphs, which have attracted most of the research and are better understood, are planar. On the other hand, regarding the applications in shape analysis, the data are what they are, and the user would not have the possibility of moving the points around to decrease the number of crossings. It is worth mentioning here that, while higher order proximity graphs were introduced and studied about twenty years ago [24, 25], there has been a renewal of interest on them, especially for low orders, as they offer a flexibility which is desirable in several applications. For example, the Delaunay triangulation (DT) is unique, while one can extract a large number of different triangulations from the 1-Delaunay graph, all of them "close" to DT, which may be preferable under some criterion (see for example the papers [16, 1] and the numerous references there).

From the viewpoint of proximity drawings, it is desirable to have a small number of crossings, and hence we study its minimum value. On the other hand, we also consider the shape analysis situation in which choosing the points is not possible, which leads to study how large the number of crossings can be, i.e., its maximum value.

For example, consider the k-nearest neighbor graph of point sets P with $|P| = n$. We introduce and study the *rectilinear crossing number* and the *worst crossing number* defined respectively as

$$\begin{aligned} \overline{cr}(k\text{-}\mathsf{NNG}(n)) &= \min_{|P|=n} \boxtimes(k\text{-}\mathsf{NNG}(P)), \\ \overline{wcr}(k\text{-}\mathsf{NNG}(n)) &= \max_{|P|=n} \boxtimes(k\text{-}\mathsf{NNG}(P)). \end{aligned}$$

We define analogous parameters for the k-relative neighborhood graph, k-$\mathsf{RNG}(P)$, in which

p_i, p_j are adjacent if the open intersection of the circles centered at p_i and p_j with radius $|p_i p_j|$ contains at most k points from P; the k-Gabriel graph, k-$\mathsf{GG}(P)$, in which p_i and p_j are adjacent if the closed circle with diameter $p_i p_j$ contains at most k points from P different from p_i, p_j; and the k-Delaunay graph, k-$\mathsf{DG}(P)$. It is well known that

$$(k+1)\text{-}\mathsf{NNG}(\mathsf{P}) \subseteq k\text{-}\mathsf{RNG}(P) \subseteq k\text{-}\mathsf{GG}(P) \subseteq k\text{-}\mathsf{DG}(P). \tag{1}$$

Notice that, when the rectilinear crossing number of a combinatorial graph is considered, we draw *the same graph* on top of different points sets, while here we study a specific kind of proximity graph on top of different point sets, but the underlying combinatorial graphs *may be different* for many of these sets. Another somehow subtle issue that deserves a specific comment is the fact that the combinatorial graph obtained from a proximity drawing may have a smaller crossing number than the rectilinear crossing number of its proximity drawing. This is clearer with an example: We prove in this paper that $\overline{cr}(1\text{-}\mathsf{DG}(n)) = n-4$; this means that 1-$\mathsf{DG}(P)$ contains at least $n-4$ crossings for any set P of n points, and that for some point set Q this number is achieved. The graph on Figure 1 (left) is the 1-Delaunay graph of its vertex set (the six shown points) and has 2 crossings; however, the combinatorial graph can be drawn on top of a different set and have only one crossing (Figure 1, right). Obviously the latter is not the 1-Delaunay graph of its vertex set.

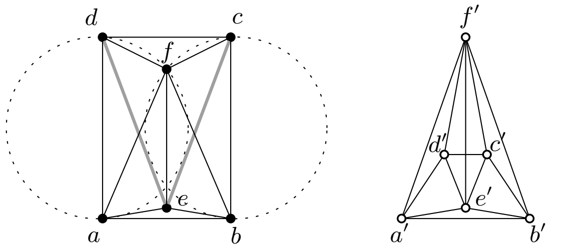

Figure 1: The graph on the left is a 1-Delaunay graph; black edges belong to 0-DG. The graph on the right is isomorphic.

A substantial part of our research focus on the 1-Delaunay graph and on the graphs k-$\mathsf{NNG}(P)$ with small k, widely used in classification scenarios, as these are the most interesting situations from the viewpoint of applications [1, 7, 10, 16]. We present these results in Subsections 2.1 and 2.2. In Subsection 2.3 we look at the number of crossings for large values of k. Throughout the paper we assume that point sets P are in *general position* in an extended sense meaning: no three points are collinear, no four points are concyclic and, for each $p \in P$, the set of its k nearest points in P is well-defined, i.e., has cardinality k, for any $k \geq 1$.

Throughout the paper we denote by $V(G)$ (respectively, $E(G)$) the set of vertices (respectively, edges) of a given graph G, and by $v(G)$ (respectively, $e(G)$) the cardinality of this set. If v is a vertex in $V(G)$, we denote by $d_G(v)$ the degree of v in G. We consider a generic set P of n points in general position, and we denote by h the size of the convex hull of P.

2 Results

Given the number of results in the paper and the length of some proofs, in this section we only state our bounds deferring the proofs to the subsequent section.

2.1 1-Delaunay graphs

In this subsection we carry out a detailed analysis of the number of crossings in a 1-Delaunay graph. We study the general case and also the particular case where all points are in convex position. Our contributions are presented in Table 1. Note that we establish the exact value of the rectilinear crossing number of the 1-Delaunay graph for both the general case and the convex case.

	general case	convex case
$\overline{cr}$	$n-4$	$6n-3\lfloor\frac{n}{2}\rfloor-19$
$\overline{wcr}$	$n^2+\Theta(n)\leq\overline{wcr}\leq 4n^2+\Theta(n)$	$\frac{n^2}{2}+\Theta(n)\leq\overline{wcr}\leq\frac{7n^2}{8}+\Theta(n)$

Table 1: 1-Delaunay graphs.

As shown in [1], the number of elements in $E(\textsf{1-DG}(P))-E(\textsf{0-DG}(P))$ is linear. Since $\textsf{0-DG}(P)$ is maximal planar, this immediately yields that every 1-Delaunay graph contains a linear number of crossings. More accurate observations lead to the following bound:

Theorem 2.1.1. *For every point set P, $\boxtimes(\textsf{1-DG}(P))\geq n-4$.*

Proposition 2.1.2. *There exists a point set Q such that $\boxtimes(\textsf{1-DG}(Q))=n-4$.*

If P is in convex position, the bounds can be strengthened:

Theorem 2.1.3. *For every set P in convex position, $\boxtimes(\textsf{1-DG}(P))\geq 6n-3\lfloor\frac{n}{2}\rfloor-19$.*

Proposition 2.1.4. *There exists a point set Q in convex position such that $\boxtimes(\textsf{1-DG}(Q))=6n-3\lfloor\frac{n}{2}\rfloor-19$.*

In principle, every pair of edges in $E(\textsf{1-DG}(P))-E(\textsf{0-DG}(P))$ might cross, so the number of crossings in $\textsf{1-DG}(P)$ could be quadratic. In the following lines we provide quadratic upper bounds on the number of crossings of $\textsf{1-DG}(P)$, and show that in some cases this parameter is indeed quadratic.

Theorem 2.1.5. *For every set of points P, $\boxtimes(\textsf{1-DG}(P))\leq 4n^2+\Theta(n)$.*

Proposition 2.1.6. *There exists a point set Q such that $\boxtimes(\textsf{1-DG}(Q))=n^2+\Theta(n)$.*

For the convex case we prove tighter bounds:

Theorem 2.1.7. *For every set of points P in convex position, $\boxtimes(\textsf{1-DG}(P))\leq 7n^2/8+\Theta(n)$.*

Proposition 2.1.8. *There exists a set of points Q in convex position such that $\boxtimes(\textsf{1-DG}(Q))=n^2/2+\Theta(n)$.*

2.2 k-nearest neighbor graphs for small values of k

We provide bounds on the rectilinear crossing number of the k-nearest neighbor graph $\textsf{k-NNG}$ for $k\leq 10$. Due to the inclusion relations satisfied by the graphs we investigate, the lower bounds also hold for the rectilinear crossing number of the other proximity graphs if we shift the value of k one unit down (see (1)).

Our results are summarized in Table 2. It is interesting noticing that, even though the lower bounds do not rely on specific properties of $\textsf{k-NNG}$ but on generic results, for many values of k we are able to construct point sets attaining these bounds.

Theorem 2.2.1. *The rectilinear crossing number of* k-NNG, *when* $k \in \{1, 2, \ldots, 10\}$, *satisfies the equalities and inequalities shown in Table 2.*

k	$\overline{cr}(k\text{-}\mathsf{NNG}(n))$
1	0
2	0
3	0
4	0, for $n \geq 14$
5	0, for $n \geq 44$
6	≤ 58, for $n \geq 39$
7	$n/2 + \Theta(1)$
8	$n + \Theta(1)$
9	$13n/6 + 50/3 \leq \overline{cr} \leq 31n/13 + \Theta(1)$
10	$10n/3 + 50/3 \leq \overline{cr} \leq 4n + \Theta(1)$

Table 2: $\overline{cr}(k\text{-}\mathsf{NNG}(n))$ for the first values of k.

As for the worst crossing number, we will give bounds for all k in Subsection 2.3, and we can improve on these only minimally for small k. Thus we omit those details.

2.3 General bounds

In this subsection we are interested in the number of crossings in the graphs under study when the value of k is large. We have derived bounds for both the rectilinear crossing number and the worst crossing number of all graphs (see Table 3). Observe that in all cases we can specify the exact order of magnitude of these parameters up to multiplicative constants.

	k-$\mathsf{NNG}(n)$	k-$\mathsf{RNG}(n)$	k-$\mathsf{GG}(n)$	k-$\mathsf{DG}(n)$
$\overline{cr} \geq$	$\frac{128}{31827}k^3n$	$\frac{128}{31827}k^3n$	$\frac{128}{31827}k^3n$	$\frac{1024}{31827}k^3n$
$\overline{cr} \leq$	$\frac{1}{9\pi^2}k^3n$	$\frac{\pi}{9(2\pi/3-\sqrt{3}/2)^3}k^3n$	$\frac{64}{9\pi^2}k^3n$	$\frac{64}{9\pi^2}k^3n$
$\overline{wcr} \geq$	$\frac{1}{3}k^3n$	$\frac{1}{3}k^3n$	$\frac{1}{4}k^2n^2$	$\frac{1}{2}k^2n^2$
$\overline{wcr} \leq$	k^3n	$9k^3n$	$3k^2n^2$	$3k^2n^2$

Table 3: Dominant terms of the general bounds. Some of the bounds only hold for "intermediate" values of k. We refer to the precise statements in the rest of the subsection.

2.3.1 Rectilinear crossing number

Our lower bounds for the rectilinear crossing numbers follow from an improved version of the crossing lemma given in [21].

Theorem 2.3.1. *If* $k \geq 13$, *then* *i)* $\overline{cr}(k\text{-}\mathsf{NNG}(n)) \geq \frac{128}{31827}k^3n$,

ii) $\overline{cr}(k\text{-}\mathsf{RNG}(n)) \geq \frac{128}{31827}k^3n$,

iii) $\overline{cr}(k\text{-}\mathsf{GG}(n)) \geq \frac{128}{31827}k^3n$.

If $6 \leq k < \frac{n}{2} - 1$, *then*

iv) $\overline{cr}(k\text{-}\mathsf{DG}(n)) \geq \frac{1024}{31827}k^3n$.

As already observed in [1], if $k \geq \frac{n}{2} - 1$, then k-$\mathsf{DG}(P)$ is the complete graph.

For the upper bounds we use a suitable construction proposed in [23]. This construction is the current asymptotically best example of a graph with fixed number of edges and minimum number of crossings. In the next proposition we show that it can be seen as a proximity graph.

Proposition 2.3.2. *If* $\omega(1) \leq k \leq o(n)$, *there exists a point set* Q *such that*

i) $\boxtimes(k\text{-}\mathsf{NNG}(Q)) \leq \frac{1}{9\pi^2}k^3n(1+o(1))$,

ii) $\boxtimes(k\text{-}\mathsf{RNG}(Q)) \leq \frac{\pi}{9(2\pi/3-\sqrt{3}/2)^3}k^3n(1+o(1))$,

iii) $\boxtimes(k\text{-}\mathsf{GG}(Q)) \leq \frac{64}{9\pi^2}k^3n(1+o(1))$,

iv) $\boxtimes(k\text{-}\mathsf{DG}(Q)) \leq \frac{64}{9\pi^2}k^3n(1+o(1))$.

2.3.2 Worst crossing number

Any upper bound on the number of edges of some higher order proximity graph can be used to produce an upper bound on its worst crossing number. For k-Delaunay graphs, it has been proved that the number of edges is at most $3(k+1)n - 3(k+1)(k+2)$ [1]. In the worst scenario, all pairs of edges might cross, so the number of crossings is no more than $\frac{9}{2}k^2n^2 + o(k^2n^2)$. In the following theorem we improve this bound:

Theorem 2.3.3. *For every point set* P, *if* $k < n/2 - 1$, *then* $\boxtimes(k\text{-}\mathsf{GG}(P)) \leq \boxtimes(k\text{-}\mathsf{DG}(P)) \leq (3k^2+6k+3)n^2 + (-6k^3 - 21k^2 - \frac{51}{2}k - \frac{21}{2})n + (3k^4 + 15k^3 + \frac{57}{2}k^2 + \frac{51}{2}k + 9)$.

The preceding bounds are tight up to a multiplicative constant:

Proposition 2.3.4. *If* $k = o(n)$, *there exists a point set* Q *such that* $\boxtimes(k\text{-}\mathsf{GG}(Q)) = k^2n^2/4 + o(k^2n^2)$.

Proposition 2.3.5. *If* $k = o(n)$, *there exists a point set* Q *such that* $\boxtimes(k\text{-}\mathsf{DG}(Q)) = k^2n^2/2 + o(k^2n^2)$.

For k-relative neighborhood graphs, it can be shown that the number of edges is bounded from above by $3kn + 3n$ (see Appendix), which yields an upper bound of $9k^2n^2 + o(k^2n^2)$ for the worst crossing number. We have proved that the order of magnitude of this parameter is lower provided that $k = o(n)$:

Theorem 2.3.6. *For every point set* P, $\boxtimes(k\text{-}\mathsf{RNG}(P)) \leq (9k^2 + 18k + 9)kn$.

Finally, the number of edges of k-NNG is no greater than kn. In this case Theorem 2.3.7 and Proposition 2.3.8 show that the worst crossing number is also cubic in k and linear in n:

Theorem 2.3.7. *For every point set* P, $\boxtimes(k\text{-}\mathsf{NNG}(P)) \leq (2k^2 - 3k + 1)kn/2$.

Proposition 2.3.8. *If* $k = o(n)$, *there exists a point set* Q *such that* $\boxtimes(k\text{-}\mathsf{RNG}(Q)) \geq \boxtimes(k\text{-}\mathsf{NNG}(Q)) = k^3n/3 + o(k^3n)$.

3 Proofs

3.1 Proofs of the results in Subsection 2.1

Let us introduce some notation. We partition the edges of the 1-Delaunay graph into two groups: we say that an edge is *blue* if it also appears in $\mathsf{0\text{-}DG}(P)$ and we say that it is *red* otherwise. We set $e_b = e\,(\mathsf{0\text{-}DG}(P))$ and $e_r = e\,(\mathsf{1\text{-}DG}(P)) - e\,(\mathsf{0\text{-}DG}(P))$. Note that a red edge p_ip_j corresponds to an element in $E(\mathsf{0\text{-}DG}(P \setminus \{p_l\}))$ for some $p_l \in P$. We say that p_ip_j is *generated* by p_l. Observe that the fact that p_ip_j is generated by p_l is equivalent to the existence of a disk through p_i and p_j containing p_l and no other point in P, which implies that p_ip_l and p_jp_l belong to $E(\mathsf{0\text{-}DG}(P))$. Thus p_ip_j is generated by at most two points. (See [1].)

In the figures of this subsection the blue edges will be represented in black and the red edges will be represented in gray.

3.1.1 Proof of Theorem 2.1.1

Since the graph $\mathsf{0\text{-}DG}(P)$ is maximal planar, each red edge induces at least one crossing in $\mathsf{1\text{-}DG}(P)$. We prove that the number of red edges in $\mathsf{1\text{-}DG}(P)$ is at least $n - 4$ (see Theorem 3.1.5). Part of our proof follows the lines of previous techniques used in [1].

Let us introduce some notation. Let H and I be the convex and interior points of P. For convenience, sets are denoted with a capital letter and their cardinalities with the corresponding lower case letter (thus h and i denote the cardinality of H and I respectively).

For a point $p \in P$, let $d_G(p)$ denote the degree of p in G, where $G = \mathsf{0\text{-}DG}(P)$, and let $d^*(p)$ be the value 2 plus the number of points of I that are in the convex hull of $P \setminus \{p\}$.

Our proof requires several remarks, already stated in [1]:

Lemma 3.1.1. *The set of red edges generated by exactly two points induces a perfect matching on the triangles of* $\mathsf{0\text{-}DG}(P)$*. Moreover, two so matched triangles are adjacent triangles of* $\mathsf{0\text{-}DG}(P)$ *in convex position.*

Lemma 3.1.2. *The number of red edges generated by an element $p \in P$ is:*

- $d_G(p) - d^*(p)$, *if* $p \in H$;
- $d_G(p) - 3$, *if* $p \in I$.

Lemma 3.1.3. *If $p_l \in I$ is in the convex hull of both $P \setminus \{p_i\}$ and $P \setminus \{p_j\}$, for some pair of points $p_i, p_j \in H$, then p_i and p_j are consecutive vertices in the convex hull of P. Furthermore, the triangle $\triangle p_ip_jp_l$ is empty, and the line through p_i and p_l separates $\triangle p_ip_jp_l$ from the rest of P, as does the line through p_j and p_l.*

Lemma 3.1.4. *Each element of I can contribute to $d^*(p_i)$ and $d^*(p_j)$ for at most two points $p_i, p_j \in H$, provided that $n \geq 5$.*

By Lemma 3.1.4, we may partition I into $I_0 \cup I_1 \cup I_2$, where I_j is the set of elements of I that contribute to d^* for exactly j points.

Notice that, if $p_l \in I_2$ contributes to d^* for p_i and p_j, then, by Lemma 3.1.3, the points p_i, p_j, and p_l together with any other point of P are not in convex position. Hence $\triangle p_ip_jp_l$ is a triangle of $\mathsf{0\text{-}DG}(P)$ that does not participate in the matching given by Lemma 3.1.1. Such a triangle is called *special* triangle.

We wish to bound the number of red edges in 1-DG(P). By Lemma 3.1.2,

$$\begin{aligned} e_r &= \sum_{p \in H} (d_G(p) - d^*(p)) + \sum_{p \in I} (d_G(p) - 3) - \xi = \\ &= 4n - 6 - (i + \sum_{p \in H} d^*(p)) - \xi, \end{aligned} \tag{2}$$

where is ξ the number of times a red edge is overcounted in the summation (which happens when two points induce the same edge). Since the set of red edges generated by the removal of two points induces a matching in the triangles of 0-DG(P), we may now introduce a new equation:

$$\xi = \frac{\triangle - i_2 - m}{2}, \tag{3}$$

where $\triangle$ is the number of triangles in 0-DG(P) (thus $\triangle = h + 2i - 2$), and m is the number of non-special triangles in 0-DG(P) not matched by a red edge generated by two points.

Substituting (3) in (2) and using that $\sum_{p \in H} d^*(p) = 2h + i_1 + 2i_2$, we obtain that

$$e_r = n - 5 + i_0 + \frac{1}{2}(h - i_2 + m).$$

Since $i_0 \geq 0$, $h - i_2 \geq 0$, and $m \geq 0$, we have that $e_r \geq n - 5$, and $e_r = n - 5$ if and only if:

$$i_0 = 0, \; h = i_2, \text{ and } m = 0. \tag{4}$$

We make some observations about the structure of P in the case where (4) is satisfied and introduce some useful notation.

Note that any point that contributes to d^* for some other point of H is in the second convex layer of P. Therefore if we assume that $i_0 = 0$, then P has exactly two convex layers, and the second one is given by the set $I = I_2 \cup I_1$. We say that each point $p_l \in I_2$ is *associated* to two points of H: those points for whom q contributes to d^*. Similarly, each point $p_l \in I_1$ is associated to the point $p_i \in H$ for which p_l contributes to $d^*(p_i)$.

With these last observations we are ready to prove a lower bound on the number of red edges.

Theorem 3.1.5. *The graph* 1-DG(P) *contains at least* $n - 4$ *red edges.*

Proof. Suppose by means of a contradiction that $e_r = n - 5$ and thus (4) holds.

We distinguish two cases.

First we assume that every edge of the convex hull of I is an edge of 0-DG(P). In this case every triangle having exactly one point of H as a vertex is entirely contained between the first and second convex layers of P; see Figure 2 (left). No two of these triangles are matched, because the four vertices of two such adjacent triangles are not in convex position. As $m = 0$ and the special triangles cannot be matched, we infer that every triangle having exactly one point of H as a vertex is matched with one triangle contained in the second convex layer of P. However, there are i triangles of the first type and $i - 2$ triangles of the second type. Thus $i_0 = 0$, $h = i_2$, and $m = 0$ cannot simultaneously hold in this case.

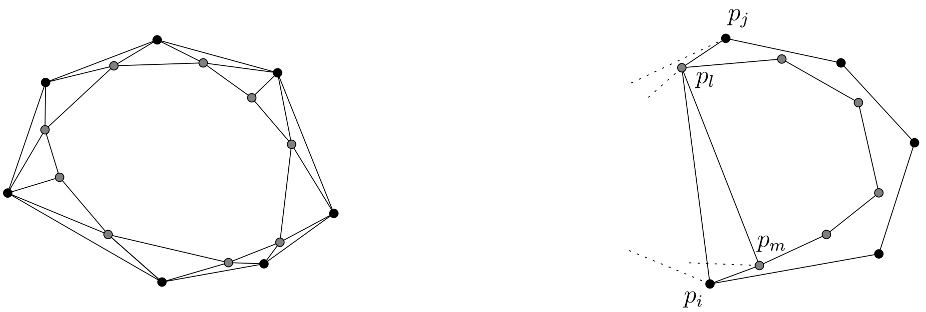

Figure 2: Part of the Delaunay triangulations of the point sets. Interior points are in gray. In the right figure, every edge of the convex hull of I is an edge of 0-DG(P). In the left figure, p_ip_l is a diagonal such that $G_1(p_ip_l)$ contains no diagonal of 0-DG(P).

Now let us suppose that there exists an edge e of the convex hull of I that is not an edge of 0-DG(P). Then there is an edge p_ip_l in 0-DG(P) crossing e such that p_i is a point of H and p_l is not associated with p_i. We call such an edge p_ip_l a *diagonal*.

For each diagonal p_ip_l, if $p_l \in H$, let $\mathrm{pol}(p_ip_l)$ be the edge p_ip_l; otherwise, let $\mathrm{pol}(p_ip_l)$ be the polygonal chain $p_ip_lp_j$, where p_j is a point of H with whom p_l is associated. In both cases $\mathrm{pol}(p_ip_l)$ is a polygonal chain joining two vertices of H. If we take the two sub-polygonal chains of the convex hull of P joining the endpoints of $\mathrm{pol}(p_ip_l)$ together with $\mathrm{pol}(p_ip_l)$, we define two closed polygonal chains that are the boundary of two bounded regions $C_1(p_ip_l)$ and $C_2(p_ip_l)$.

The regions $C_1(p_ip_l)$ and $C_2(p_ip_l)$ define two non-empty subsets $P_1(p_ip_l)$ and $P_2(p_ip_l)$, the points of P that lie at the interior or at the boundary of $C_1(p_ip_l)$ and $C_2(p_ip_l)$, respectively. Without loss of generality, assume that $P_1(p_ip_l)$ is the smallest of the two sets. Let $G_1(p_ip_l)$ and $G_2(p_ip_l)$ be the subgraphs of 0-DG(P) induced by $P_1(p_ip_l)$ and $P_2(p_ip_l)$.

Let p_ip_l be a diagonal in 0-DG(P) such that $G_1(p_ip_l)$ contains no diagonal of 0-DG(P). Observe that p_l is a point of I. Among all points of H with whom p_l is associated, let p_j be the one that yields the smallest possible value for $|P_1(p_ip_l)|$. Now let $I' := I \cap P_1(p_ip_l)$ and $H' := H \cap P_1(p_ip_l)$.

The intersection of the convex hulls of I and I' is a convex polygonal chain Q_2 that has I' as its vertex set and p_l as an endpoint. Since the only diagonal of 0-DG(P) in $G_1(p_ip_l)$ is p_ip_l, Q_2 is a subgraph of $G_1(p_ip_l)$ (see Figure 2, right). Let p_m denote the endpoint of Q_2 different from p_l. The edge p_ip_l is a side of some triangle of $G_1(p_ip_l)$; since there are no diagonals of 0-DG(P) in $G_1(p_ip_l)$ except for p_ip_l, the third vertex of this triangle is p_m. Thus $G_1(p_ip_l)$ contains a triangulation of I' as subgraph.

Let T_h be the set of triangles of $G_1(p_ip_l)$ consisting of an edge of Q_2 and a point in H', and T_i be the set of triangles of $G_1(p_ip_l)$ with all their vertices in I'. It is not difficult to see that any triangle in T_h that participates in the matching is matched with a triangle in T_i. Since $|T_h| = i' - 1$ and $|T_i| = i' - 2$, we conclude that m is greater than zero also in this case. □

3.1.2 Proof of Proposition 2.1.2

We start with $n-2$ points in a vertical segment, denoted from top to bottom by $q_1, q_2, \ldots, q_{n-2}$. We add one point to the left of this group, and one point to the right, as in Figure 3 (left). Then we slightly move the even points $q_2, q_4, \ldots$ to the right, and the odd points $q_3, q_5, \ldots$ to the left.

The only red edges in 1-DG(Q) are q_iq_{i+2}, for $i = 1, 2, \ldots, n-4$. No pair of such edges crosses, and each of them creates exactly one crossing with 0-DG(Q). Thus the number of crossings of

1-$\mathsf{DG}(Q)$ is $n-4$.

3.1.3 Proof of Theorem 2.1.3

Let $p_1, \ldots, p_n$ denote the points in P in clockwise order. Note that all edges of type $p_i p_{i+2}$ are in 1-$\mathsf{DG}(P)$, and that the total number of crossings between two edges of this family is n. Let G' be the graph obtained from 1-$\mathsf{DG}(P)$ by removing these edges and the ones in the convex hull of P. Since $e_r \geq 2n - \lfloor \frac{n}{2} \rfloor - 5$ (see [1]), G' contains at least $2n - \lfloor \frac{n}{2} \rfloor - 8$ edges. Each of them induces two crossings with the edges that have been removed.

Let G'_p be a maximal planar subgraph of G'. It is easy to see that G'_p contains at most $n-5$ edges. Thus there are at least $n - \lfloor \frac{n}{2} \rfloor - 3$ edges in G' but not in G'_p, each of which induces at least one crossing with an edge of G'_p.

Adding everything up, the graph 1-$\mathsf{DG}(P)$ has no less than $6n - 3\lfloor \frac{n}{2} \rfloor - 19$ crossings.

3.1.4 Proof of Proposition 2.1.4

Consider two horizontal lines such that each point in one line has a counterpart in the other line with the same abscissa. Add one point to the left of both lines such that its ordinate is the average of the ordinates of the lines. If the positions of the points are carefully chosen, it is possible to perturb them so that the point set is in convex position and 1-$\mathsf{DG}(Q)$ contains only the edges drawn in Figure 3 (middle). Easy calculations show that, in this case, the number of crossings of the graph is $6n - 3\lfloor \frac{n}{2} \rfloor - 19$.

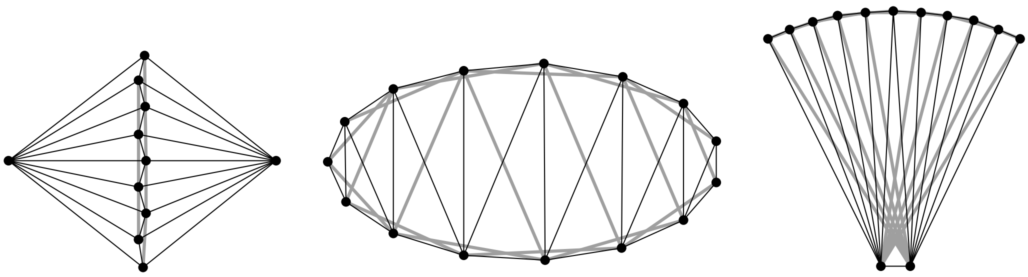

Figure 3: Left: point set whose 1-Delaunay graph has $n-4$ crossings. Middle: point set in convex position whose 1-Delaunay graph has $6n - 3\lfloor \frac{n}{2} \rfloor - 19$ crossings. Right: point set in convex position whose 1-Delaunay graph has $n^2/2 + \Theta(n)$ crossings.

3.1.5 Proof of Theorem 2.1.5

Theorem 2.3.3 says that at most $12n^2 - 63n + 81$ crossings are present in 1-$\mathsf{DG}(P)$. In this proof we improve the dominant term of this bound to $4n^2$.

A crossing in 1-$\mathsf{DG}(P)$ is caused either by a red edge and a blue one or by a pair of red edges. We denote the cardinal number of the first and second sets of crossings by $r \otimes b$ and $r \otimes r$, respectively. We derive upper bounds for $r \otimes b$ and $r \otimes r$.

The bound for $r \otimes b$ is given in Lemma 3.1.9 and requires several technical lemmas and observations:

Observation 3.1.6. Let $p_i p_j$, $p_l p_m$ be two crossing edges. Either every circle through p_i, p_j contains p_l or p_m, or every circle through p_l, p_m contains p_i or p_j.

Lemma 3.1.7. *If u, v, w are three vertices of a planar graph G on n vertices, then $d_G(u)+d_G(v)+d_G(w) \leq 2n+2$.*

Proof. Let $V'(G)$ be the set of vertices in $V(G) \setminus \{u, v, w\}$ that are adjacent to u, v and w. Then $d_G(u) + d_G(v) + d_G(w) \leq 3|V'(G)| + 2(n - 3 - |V'(G)|) + 6 = 2n + |V'(G)|$. Since the graph $K_{3,3}$ is not planar, we have that $|V'(G)| \leq 2$. □

Lemma 3.1.8. *Let G be a graph on n vertices. If G is a plane triangulation, then $\sum_{v \in V(G)} d_G^2(v) \leq 2n^2 + 33n$.*

Proof. We prove the lemma by induction on the order of the graph. The small cases are trivial. We next proceed to the inductive step.

Let us first assume that there exists a vertex v' not in the external face having degree three, four or five. Let G' be a graph containing all the edges in $G \setminus v'$ and where the face bounded by the neighbors of v' in G has been triangulated. Let $w_1, w_2, \ldots, w_I$ be the vertices in $V(G)$ such that $d_{G'}(w_i) = d_G(w_i) - 1$. Note that: if $d_G(v') = 3$, then $I = 3$; if $d_G(v') = 4$, then $I = 2$; and, if $d_G(v') = 5$, then $I = 2$. For any $v \in V(G)$, $v \neq v, w_1, \ldots, w_I$, we have that $d_G(v) \leq d_{G'}(v)$. Then

$$\begin{aligned}
\sum_{v \in V(G)} d_G^2(v) &= \sum_{v \neq v', w_i} d_G^2(v) + d_G^2(v') + \sum_{i=1,\ldots,I} d_G^2(w_i) \leq \\
&\leq \sum_{v \neq v'} d_{G'}^2(v) + 2 \sum_{i=1,\ldots,I} d_{G'}(w_i) + d_G^2(v') + I \leq \\
&\leq \sum_{v \neq v'} d_{G'}^2(v) + 2 \sum_{i=1,\ldots,I} d_{G'}(w_i) + 27.
\end{aligned}$$

By the induction hypothesis and Lemma 3.1.7,

$$\sum_{v \in V(G)} d_G^2(v) \leq 2(n-1)^2 + 33(n-1) + 2(2n+2) + 27 = 2n^2 + 33n.$$

Now suppose that all the interior vertices have degree at least six (or there are no interior vertices). Let H be the set of vertices in the external face and let $h = |H|$. By the handshaking lemma, $\sum_{v \in H} d(v) \leq 4h - 6$. Consequently, there exists a vertex in the external face having degree three or two, and the same strategy can be used to prove the inequality. □

Lemma 3.1.9. *For every set of points P, $r \otimes b \leq n^2 + \Theta(n)$.*

Proof. If there is a crossing between a red edge r and a blue edge $b = p_i p_j$, then, by Observation 3.1.6, r is generated by p_i or p_j (or both). We assign the crossing to this point (or to any of them if r is generated by both).

Next we bound the number of crossings that may be assigned to some point $p \in P$.

First assume that p is not in the convex hull of P. Let $e_k = pq_k$ be a blue edge incident to p; we want to know how many edges in $\mathsf{0\text{-}DG}(P \setminus \{p\})$ it may cross. Consider the triangulation T constituted by the cycle connecting the neighbors of p in $\mathsf{0\text{-}DG}(P)$ and the edges generated by p (see Figure 4, left). Let T_p be the triangle containing p and T_{q_k} be the triangle incident to q_k that is traversed by e_k ($T_{q_k} = T_p$ if q_k is a vertex of T_p). Observe that the number of edges in $\mathsf{0\text{-}DG}(P \setminus \{p\})$ that e_k crosses correspond to the distance between T_{q_k} and T_p in the dual graph of T. Observe also that, if q_k and q_l are two different vertices that are adjacent to p in $\mathsf{0\text{-}DG}(P)$ and are not vertices of T_p, we have that $T_{q_k} \neq T_{q_l}$. Then it is easy to see that the configuration of the dual graph maximizing the sum of distances between T_{q_k} and T_p, for all q_k neighbor of p in $\mathsf{0\text{-}DG}(P)$ (and not vertex of T_p), is a tree rooted at T_p. Consequently, at most $\sum_{\nu=1}^{d_G(p)-3} \nu$ crossings (where $G = \mathsf{0\text{-}DG}(P)$) are assigned to p.

Next suppose that p is a vertex of the convex hull of P. Let $q_1, q_2, \ldots, q_{d_G(p)}$ be the neighbors of p in $\mathsf{0\text{-}DG}(P)$ in radial order around p ($G = \mathsf{0\text{-}DG}(P)$). Let q_τ be the first point that belongs to the convex hull of $P \setminus \{p\}$ but does not belong to the convex hull of P (if there is not such q_τ, we set $q_\tau = q_{d_G(p)}$). Let us look at the triangulation of the polygon $pq_1q_2 \ldots q_\tau p$ given by the red edges. In order to determine the number of edges in $\mathsf{0\text{-}DG}(P \setminus \{p\})$ that an edge pq_k ($k \in \{2, 3, \ldots, \tau - 1\}$) crosses we can use the same argument as before, except that in this case T_p is defined as the triangle having p as a vertex. Next we can look at the next point that belongs to the convex hull of $P \setminus \{p\}$ but does not belong to the convex hull of P (let us denote it by q_ι) and apply the same argument to the edges of the polygon $pq_\tau q_{\tau+1} \ldots q_\iota p$. We proceed in this way until we reach $q_{d_G(p)}$. Then it is not difficult to see that the number of crossing that may be assigned to p is less than or equal to $\sum_{\nu=1}^{d_G(p)-2} \nu$.

Now the result follows from Lemma 3.1.8.

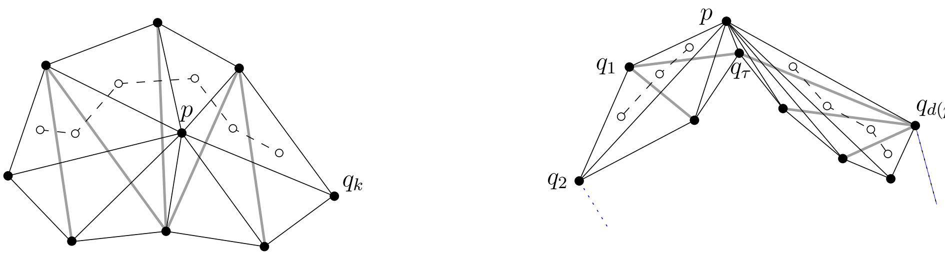

Figure 4: Crossings assigned to p. In the right figure, p is an interior point of P; in the left figure, it belongs to the convex hull of P. The dashed edges correspond to the dual graphs of the triangulations given by the red edges.

□

Next we give an upper bound for $r \otimes r$, which, combined with the bound just seen, completes the proof of Theorem 2.1.5.

Lemma 3.1.10. *For every set of points P, $r \otimes r \leq 3n^2 + \Theta(n)$.*

Proof. Let the *red crossing graph* be the graph whose vertices are the red edges of $\mathsf{1\text{-}DG}(P)$ and where two vertices are adjacent if their corresponding edges cross. First we will prove that $e_r \leq 3n - h - 6$, that is, that the red crossing graph has no more than $3n - h - 6$ vertices. Afterwards we will see that every red crossing graph has no 4-clique. Then we can apply Turán's theorem [27], which states that any K_{r+1}-free graph on m vertices has at most $(1 - \frac{1}{r})\frac{m^2}{2}$ edges. This yields the result.

First we bound the number of vertices of the red crossing graph.

Let H be the set of points in the convex hull of P, I be the set of interior points of P, and I_j ($j \in \{0, 1, 2\}$) be the set of interior points of P appearing in the convex hull of $P \setminus \{p\}$ for j distinct points $p \in H$. We denote the cardinal number of these sets by the corresponding lower case letter. For $p \in H$, let $d^*(p)$ be the value 2 plus the number of points of the convex hull of $P \setminus \{p\}$

which are not vertices of the convex hull of P. Finally, let ξ be the number of red edges that are generated by two distinct points of P. We have already seen that

$$e_r = 4n - 6 - (i + \sum_{p \in H} d^*(p)) - \xi.$$

Using that $\sum_{p \in H} d^*(p) = 2h + i_1 + 2i_2$, we obtain that

$$e_r = 2n - 6 + i_0 - i_2 - \xi.$$

Substituting $i_0 \leq n - h$, $i_2 \geq 0$, and $\xi \geq 0$, we can conclude

$$e_r \leq 3n - h - 6.$$

Next we prove that the red crossing graph is K_4-free.

By Observation 3.1.6, if two red edges $p_i p_j$, $p_l p_m$ cross, either every circle through p_i, p_j contains p_l or p_m, or viceversa. If every circle through p_i, p_j contains p_l or p_m, we say that $p_l p_m$ *constrains* $p_i p_j$.

Let $p_i p_j$, $p_l p_m$, and $p_s p_t$ be three pairwise crossing red edges. Since all the endpoints of these edges are distinct, every edge can only be constrained by one of the other two. Thus, without loss of generality, we can assume that $p_l p_m$ constrains $p_i p_j$, $p_s p_t$ constrains $p_l p_m$, and $p_i p_j$ constrains $p_s p_t$. Now suppose that there exists a red edge $p_u p_v$ that crosses $p_i p_j$, $p_l p_m$, and $p_s p_t$. Then $p_u p_v$ must be constrained by $p_i p_j$, $p_l p_m$, and $p_s p_t$, which is impossible because at least one circle through p_u, p_v only contains one point from P. □

3.1.6 Proof of Proposition 2.1.6

We use the construction described in Proposition 2.3.5 for the particular case $k = 1$. The number of crossings involving two points from the middle group and either two points from the upper group or two points from the lower group is $2\binom{n-4}{2}$.

3.1.7 Proof of Theorem 2.1.7

First of all we need a technical result.

Lemma 3.1.11. *If G is a graph such that $v(G) = n$ and $e(G) \leq n - 5$, then $\sum_{v \in V(G)} d_G^2(v) \leq e^2(G) + 3e(G)$.*

Proof. The proof of the lemma is by induction on the order of the graph. As in the proof of Lemma 3.1.8, the small cases are trivial, so we proceed to the inductive step.

Observe that there exists a vertex v' in the graph having degree zero or one. We distinguish two cases.

First assume that v' has degree zero. If G has no edges, the result trivially holds. Otherwise, let uw be an edge of G and let G' be defined as the graph $G \setminus v' \setminus uw$. We have

$$\begin{aligned}
\sum_{v \in V(G)} d_G^2(v) &= \sum_{v \neq v', u, w} d_G^2(v) + d_G^2(v') + d_G^2(u) + d_G^2(w) = \\
&= \sum_{v \neq v', u, w} d_{G'}^2(v) + (d_{G'}(u) + 1)^2 + (d_{G'}(w) + 1)^2 = \\
&= \sum_{v \in V(G')} d_{G'}^2(v) + 2((d_{G'}(u) + d_{G'}(w)) + 2
\end{aligned}$$

Applying the induction hypothesis and using the fact that $d_{G'}(u) + d_{G'}(w) \leq e(G') + 1$,

$$\sum_{v \in V(G)} d_G^2(v) \quad \leq \quad (e(G)-1)^2 + 3(e(G)-1) + 2e(G) + 2 = e^2(G) + 3e(G).$$

Next suppose that v' has degree one. Let u be the vertex adjacent to v' in G and G' be the graph $G \setminus v'$. Then,

$$\begin{aligned}
\sum_{v \in V(G)} d_G^2(v) &= \sum_{v \neq v', u} d_G^2(v) + d_G^2(v') + d_G^2(u) = \sum_{v \neq v', u} d_{G'}^2(v) + 1 + (d_{G'}(u) + 1)^2 = \\
&= \sum_{v \in V(G')} d_{G'}^2(v) + 2d_{G'}(u) + 2 \leq (e(G)-1)^2 + 3(e(G)-1) + 2(e(G)-1) + 2 \leq \\
&\leq e^2(G) + 3e(G).
\end{aligned}$$

□

Now we are ready to prove Theorem 2.1.7.

If P is in convex position, then $e_b = 2n - 3$. Since, in general, $e_r \leq 3n - h - 6$ (see the proof of Lemma 3.1.10), in the convex case we have that $e_r \leq 2n - 6$.

Let p_i, p_{i+1}, and p_{i+2} be three consecutive points in the convex hull of P. Let us suppose that we momentarily remove from $\mathsf{1\text{-}DG}(P)$ the edges $p_i p_{i+1}$, $p_{i+1} p_{i+2}$, and $p_i p_{i+2}$ for all i. Let $e_{b'}$ and $e_{r'}$ respectively be the number of blue and red edges in $\mathsf{1\text{-}DG}(P)$ after these removals. It is not difficult to see that $n/2 - 3 \leq e_{b'} \leq n - 5$ and $e_{r'} \leq 2n - 9 - e_{b'}$.

For all i, the edges $p_i p_{i+1}$ are not involved in any crossing. The edges of the form $p_i p_{i+2}$ participate in a total number of at most $5n - 18$ crossings, as pairs of edges of this type generate n crossings, and each of the (at most) $2n - 9$ remaining edges in $\mathsf{1\text{-}DG}(P)$ induces two crossings with them. Let $r' \otimes r'$ denote the number of crossings between two red edges that are not of the form $p_i p_{i+2}$. Then

$$\boxtimes\, (\mathsf{1\text{-}DG}(P)) \leq r \otimes b + r' \otimes r' + 5n - 18.$$

Let $G = \mathsf{0\text{-}DG}(P)$ and G' be the graph on P consisting of the edges of G not of the form $p_i p_{i+1}$ or $p_i p_{i+2}$. As we have seen in Lemma 3.1.9,

$$r \otimes b \leq \sum_{p \in P} \sum_{\nu=1}^{d_G(p)-2} \nu = \frac{1}{2} \sum_{p \in P} (d_G^2(p) - 3d_G(p) + 2).$$

Notice that, for all $p \in P$, $d_G(p) = d_{G'}(p) + 4$. Hence

$$r \otimes b \leq \frac{1}{2} \sum_{v \in P} (d_{G'}^2(v) + 5d_{G'}(v) + 6).$$

By Lemma 3.1.11,

$$r \otimes b \leq \frac{e_{b'}^2}{2} + \frac{13 e_{b'}}{2} + 3n.$$

Next we bound $r' \otimes r'$. Recall that $e_{r'} \leq 2n - 9 - e_{b'}$. Following the same argument as in the proof of Lemma 3.1.10, we obtain

$$r' \otimes r' \leq \frac{(2n - 9 - e_{b'})^2}{3}.$$

Thus, putting everything together,

$$\boxtimes(\text{1-DG}(P)) \le \frac{5e_{b'}^2}{6} + \left(\frac{25}{2} - \frac{4n}{3}\right) e_{b'} + \left(\frac{4n^2}{3} - 4n + 9\right) =: f(e_{b'}).$$

For n large enough, the maximum value of the function $f(e_{b'})$ in the domain $[n/2-3, n-5]$ is achieved in the lower extreme of the interval and is equal to $7n^2/8 + 15n/4 - 21$. This completes the proof.

3.1.8 Proof of Proposition 2.1.8

Consider a set of $n-2$ points on a circle together with 2 points close to its center, as in Figure 3 (right). In the graph 1-DG(Q) each point in the circular chain is adjacent to both central points. Therefore the number of crossings of 1-DG(Q) is greater than $\binom{n-2}{2}$.

3.2 Proof of the results in Subsection 2.2

In this subsection we prove the results in Table 2. The arrows in the figures have been suppressed for the sake of readability.

Proposition 3.2.1. *For any* $n \ge 2$, $\overline{cr}(\text{1-NNG}(n)) = 0$.

Proof. For any n-point set P, the graph 1-NNG(P) is plane, so it has no crossings. □

Proposition 3.2.2. *For any* $n \ge 3$, $\overline{cr}(\text{2-NNG}(n)) = 0$.

Proof. Let Q be the set of vertices of a slightly perturbed (so that no four points are concyclic) regular n-gon. Then, in 2-NNG(Q), each vertex is adjacent to its two contiguous vertices in the boundary of the polygon. Thus 2-NNG(Q) is a plane graph. □

Proposition 3.2.3. *For any* $n \ge 4$, $\overline{cr}(\text{3-NNG}(n)) = 0$.

Proof. Consider the examples of plane 3-NNG(Q) for $|Q| = 4, 5, 6, 7$ in Figure 5. Let Q_i ($i \in \{4,5,6,7\}$) denote the example with i points. If $n = 4l + j$, with $l \in \mathbb{N}$ and $j \in \{0,1,2,3\}$, we construct an n-point set made up of $l-1$ copies of Q_4 and one copy of Q_{4+j}. If these clusters are far enough from each other, the three nearest neighbor graph of the resulting set of points do not contain edges whose endpoints belong to two different clusters. Consequently, the graph is plane.

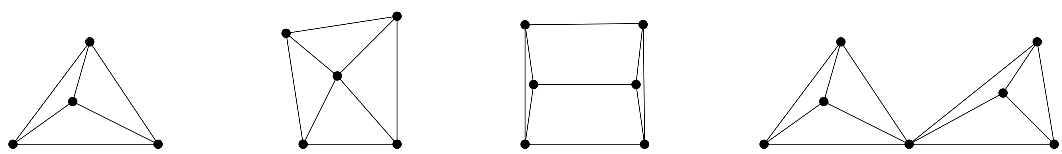

Figure 5: Point sets of cardinality 4, 5, 6 and 7 whose 3-NNG is plane.

□

Proposition 3.2.4. *For any* $n \ge 14$, $\overline{cr}(\text{4-NNG}(n)) = 0$.

Proof. If n is even, first we place the vertices of a (slightly perturbed) regular $n/2$-gon. Afterwards we add an interior $n/2$-gon such that each vertex is very close to the midpoint of one of the edges of the exterior polygon (see Figure 6, left). If $n \geq 8$, the four nearest neighbor graph of this set of points contains the boundaries of both polygons and the edges connecting each point in the exterior polygon to its two closest points in the interior polygon.

If n is odd, consider the previous construction with $n-1$ points and add a new point close to the center. If $n \geq 15$, the four nearest neighbor graph is augmented by only four new edges, namely, the ones connecting the new point to its four nearest neighbors, which are all in the interior polygon (see Figure 6, right). Hence no crossing is created.

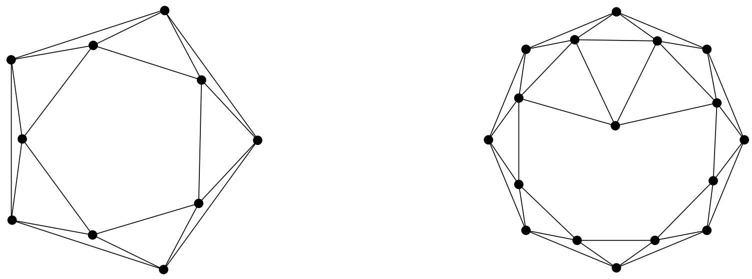

Figure 6: Point sets of cardinality 10 and 17 whose 4-NNG is plane.

$\square$

Proposition 3.2.5. *For any $n \geq 44$, $\overline{cr}(\textsf{5-NNG}(n)) = 0$.*

Proof. We start considering values of n of the form $n = 4l$, with $l \geq 13$. We place the following four groups of l points ($i \in \{1, 2, \ldots, l\}$):

$$\begin{aligned}
p_i &= \frac{1}{2\sin(\pi/l)}\left(\cos\left(\frac{2\pi i}{l}\right), \sin\left(\frac{2\pi i}{l}\right)\right), \\
q_i &= \left(\frac{1}{2\tan(\pi/l)} + \frac{\sqrt{3}}{2}\right)\left(\cos\left(\frac{2\pi i}{l} + \frac{\pi}{l}\right), \sin\left(\frac{2\pi i}{l} + \frac{\pi}{l}\right)\right), \\
r_i &= \frac{1}{2\sin(\pi/l)(1 - 2\sin(\pi/l))}\left(\cos\left(\frac{2\pi i}{l}\right), \sin\left(\frac{2\pi i}{l}\right)\right), \\
s_i &= \left(\frac{1}{2\tan(\pi/l)} + \frac{\sqrt{3}}{2}\right)(1 + 2\sin(\pi/l))\left(\cos\left(\frac{2\pi i}{l} + \frac{\pi}{l}\right), \sin\left(\frac{2\pi i}{l} + \frac{\pi}{l}\right)\right).
\end{aligned}$$

These points correspond to four regular and concentric l-gons with increasing radius (see Figure 7). Easy calculations show that $|p_i p_{i+1}| = |p_{i+1} q_i| = |q_i p_i| = 1$, $|p_i r_i| = |r_i r_{i+1}|$ and $|q_i q_{i+1}| = |q_i s_i|$. In order to break the last two equalities we slightly decrease the radius of the third and fourth polygons. We also perturb the points to reach a general position.

The five nearest neighbor graph of the resulting set of points has the edges shown in Figure 7. In particular, it is plane. If $l < 13$, the adjacencies change and the graph contains several crossings. However, for $l = 11$ and $l = 12$ these crossings can be removed by decreasing a bit the radius of the third circle. So we have proved that for any $n \geq 44$, $n \equiv 0 \pmod 4$, there exist an n-point set Q_n whose five nearest neighbor graph is plane.

If $n = 4l + j$, with $l \geq 11$ and $j \in \{1, 2, 3\}$, we can add j points close to the center of the set Q_{4l} in such a way that the five nearest neighbor graph remains plane.

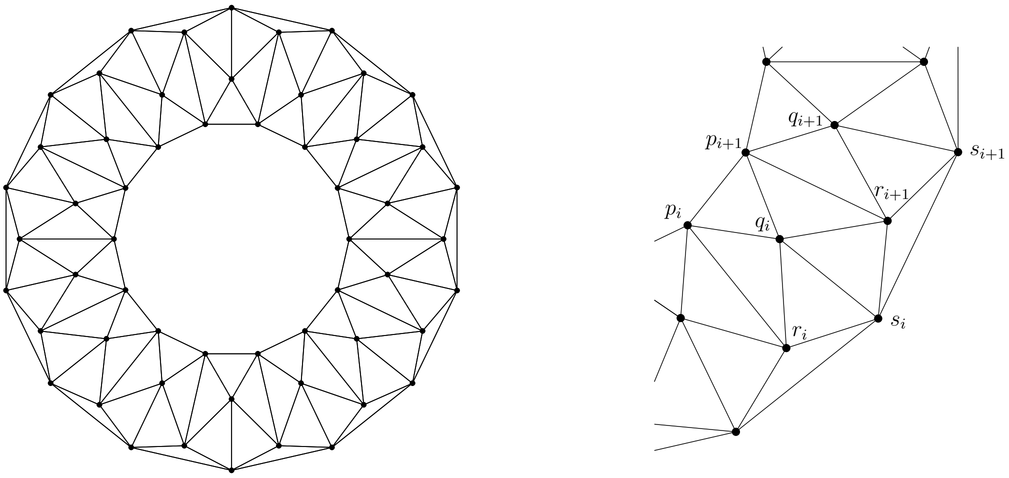

Figure 7: Set of 56 points whose 5-NNG is plane.

□

Proposition 3.2.6. *For any $n \geq 39$, $\overline{cr}(6\text{-NNG}(n)) \leq 58$.*

Proof. Let us first assume that $n = 13l$, with $l \geq 3$.

Consider a group of l regular and concentric 13-gons $R_1, R_2, \ldots, R_l$. The polygon R_i, for $i \in \{2, 3, \ldots, l-1\}$, is rotated by an angle of $\pi/13$ with respect to the polygon R_{i-1}, while R_l is rotated by an angle slightly larger than $\pi/13$ with respect to R_{l-1} to break ties. The radius of R_1 is 0.9, and the radius of R_i for $i > 2$ is 1.386^{i-1}. The points are perturbed so that they are in general position. See Figure 8.

Regardless of the value of l, the six nearest neighbor graph of this set of points has 52 crossings, as the crossings only involve vertices from R_1, R_{l-2}, R_{l-1} and R_l. This settles the problem for values of n that are multiple of 13.

If $n = 13l + j$, with $l \geq 3$ and $j \in \{1, 2, \ldots, 12\}$, we add j consecutive points of the polygon R_{l+1}. The six nearest neighbor graph of the new point set has 6 extra crossings. □

Proposition 3.2.7. *For any $n \geq 11$,*

i) $\overline{cr}(7\text{-NNG}(n)) \geq \frac{n}{2} + 6$,

ii) $\overline{cr}(8\text{-NNG}(n)) \geq n + \frac{50}{3}$,

iii) $\overline{cr}(9\text{-NNG}(n)) \geq \frac{13n}{6} + \frac{50}{3}$,

iv) $\overline{cr}(10\text{-NNG}(n)) \geq \frac{10n}{3} + \frac{50}{3}$,

Proof. For every set of points P, the number of edges of k-NNG(P) is at least $kn/2$, since each vertex has degree k or greater. Now the first bound follows from the well-known fact that, for any graph G, its crossing number satisfies that $cr(G) \geq e(G) - (3v(G) - 6)$. The remaining bounds are a corollary of the following result:

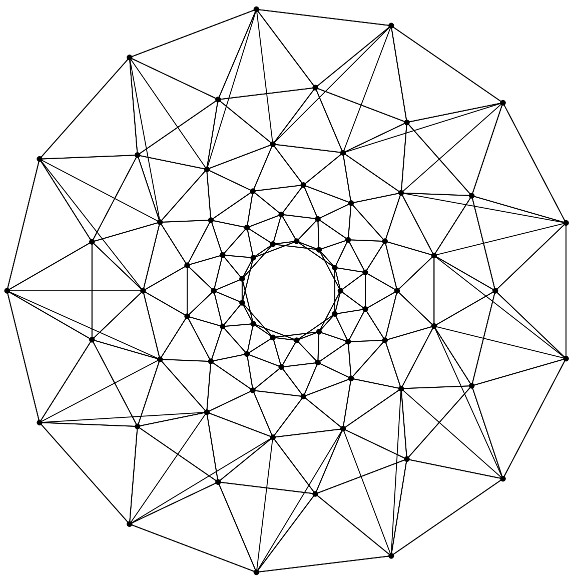

Figure 8: Set of 78 points whose 6-NNG has 52 crossings.

Theorem 3.2.8. [21] *The crossing number of any graph G with $v(G) \geq 3$ vertices and $e(G)$ edges satisfies*

$$cr(G) \geq \frac{7}{3}e(G) - \frac{25}{3}(v(G) - 2).$$

□

Proposition 3.2.9. *For any $n \geq 8$, $\overline{cr}(\text{7-NNG}(n)) \leq \frac{n}{2} + \Theta(1)$.*

Proof. If $n \leq 24$, the result is trivial.

If $n = 25l$ with $l \geq 1$, we place l regular and concentric 25-gons $R_1, R_2, \ldots, R_l$ (see Figure 9, left). The polygons R_i such that $i = 4j + 1$ or $i = 4j + 2$ for some $j \geq 0$ have all the same orientation, while the remaining polygons are rotated by an angle of $\pi/25$ with respect to them. For all i, the radius of R_i is 1.27^i. The points are perturbed to attain general position.

Ignoring some crossings that occur near the boundaries, the seven nearest neighbor graph of this point set contains $n/2$ crossings (or $(n - 25)/2$, depending on the parity of l), because the crossings only take place between consecutive 25-gons of the form R_{2j+1}, R_{2j+2}, and, for each pair, the number of such crossings is 25. The crossings near the boundaries only contribute an additive factor of constant size.

Finally, if $n = 25l + j$, with $l \geq 1$ and $j \in \{1, 2, \ldots, 24\}$, we add j consecutive points of the polygon R_{l+1}. This only adds a constant number of extra crossings.

□

Proposition 3.2.10. *For any $n \geq 9$, $\overline{cr}(\text{8-NNG}(n)) \leq n + \Theta(1)$.*

Proof. We also use concentric polygons. For constant values of n the bound is trivial, and for $n = 26l + j$ we use the same strategy as in previous cases, so here we focus on the case where $n = 26l$.

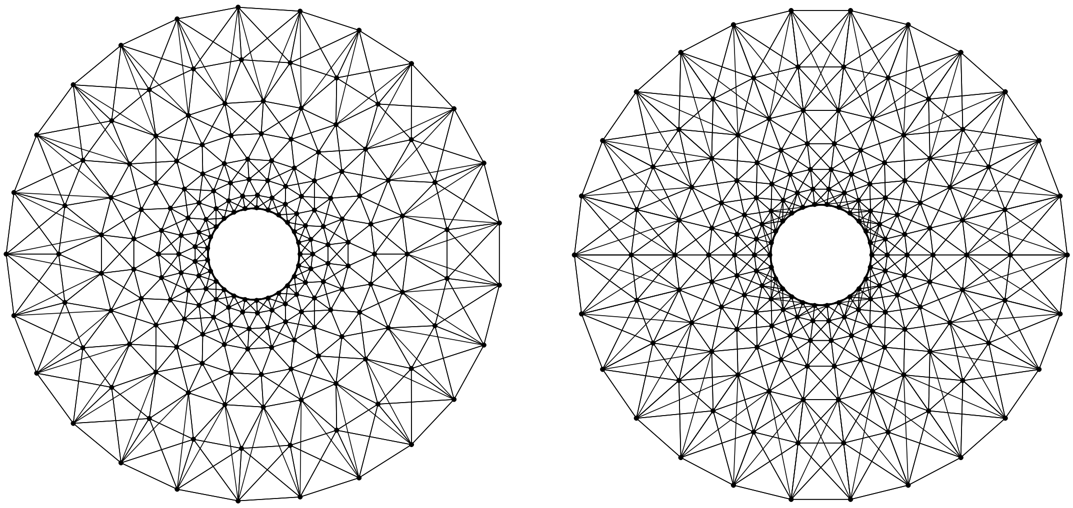

Figure 9: Left: point set whose 7-NNG has $n/2 + \Theta(1)$ crossings. Right: point set whose 8-NNG has $n + \Theta(1)$ crossings.

We consider l regular and concentric 26-gons $R_1, R_2, \ldots, R_l$ with the same orientation (see Figure 9, right). For all i, the radius of R_i is 1.3^i. The points are infinitesimally perturbed.

The eight nearest neighbor graph of this point set has $n+\Theta(1)$ crossings. The linear term comes from the fact that in the region between any pair of consecutive 26-gons there are 26 crossings. The constant term comes from some additional crossings that take place near the boundaries of the point set. $\square$

Proposition 3.2.11. *For any* $n \geq 10$, $\overline{cr}(\mathsf{9\text{-}NNG}(n)) \leq 31n/13 + \Theta(1)$.

Proof. We propose the construction in Figure 10. A careful analysis of the drawing yields that the nine nearest neighbor graph of the point set has $31n/13+\Theta(\sqrt{n})$ crossings. The term $\Theta(\sqrt{n})$ comes from crossings that take place near the boundary of the point set. Since the 9 nearest neighbors of each point are well-defined, the point positions can be slightly perturbed without modifying the set of nearest neighbors of each point. Thus we can rearrange the points in circular strips, where each strip contains exactly the minimum number of points ensuring that the adjacencies in the nine nearest neighbor graph do not change. This reduces the number of crossings to $31n/13+\Theta(1)$. We omit further details due to the high complexity of the point set.

$\square$

Proposition 3.2.12. *For any* $n \geq 11$, $\overline{cr}(\mathsf{10\text{-}NNG}(n)) \leq 4n + \Theta(1)$.

Proof. Our construction is shown in Figure 11. It can be seen that the ten nearest neighbor graph of the point set contains $4n+\Theta(\sqrt{n})$ crossings. Using the same strategy as in the previous example, we can modify the construction to obtain a new set of points whose 10-NNG has $4n+\Theta(1)$ crossings.

$\square$

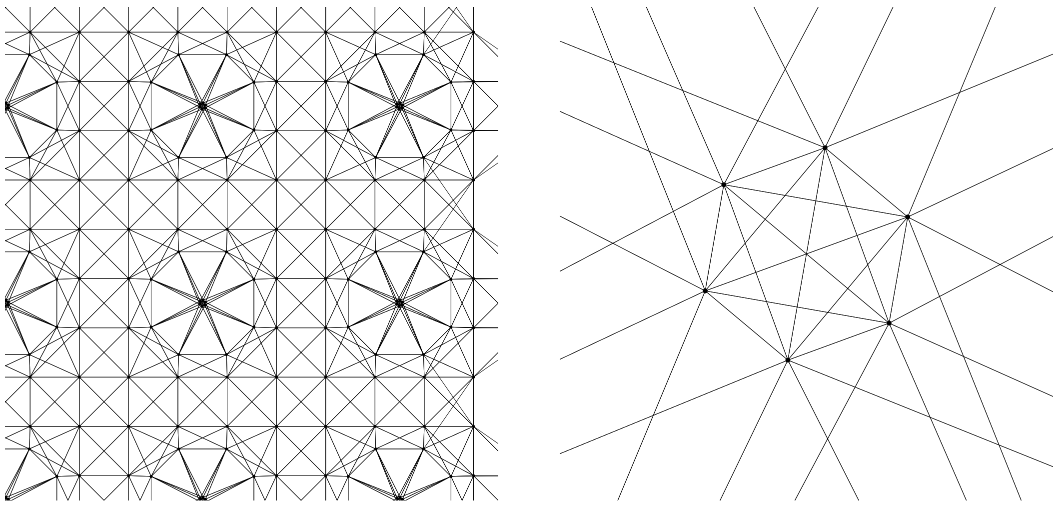

Figure 10: Left: set of points whose 9-NNG has $31n/13 + \Theta(\sqrt{n})$ crossings. Right: zoom of the figure on the left.

3.3 Proofs of the results in Subsection 2.3

3.3.1 Proof of Theorem 2.3.1

For every set of points P, the number of edges of k-$\mathsf{NNG}(P)$ is no less than $kn/2$. The graphs k-$\mathsf{RNG}(P)$ and k-$\mathsf{GG}(P)$ contain all edges present in k-$\mathsf{NNG}(P)$, so they also have at least $kn/2$ edges.

A stronger lower bound is known for the graph k-$\mathsf{DG}(P)$. If $k < \frac{n}{2} - 1$, then the number of edges of k-$\mathsf{DG}(P)$ is at least $(k+1)n$ (see [1]).

Now the bounds on the number of crossings follow from the next theorem:

Theorem 3.3.1. [21] *The crossing number of any graph G such that $e(G) \geq \frac{103}{16} v(G)$ satisfies $cr(G) \geq \frac{1024}{31827} \frac{e^3(G)}{v^2(G)}$.*

3.3.2 Proof of Proposition 2.3.2

We use the following result in [23]. We note that, instead of $\pi/9$, the incorrect coefficient $2\pi/27$ was originally reported. The correct coefficient was later reported in [21].

Proposition 3.3.2. [23] *Let $\omega(1) \leq d \leq o(\sqrt{n})$. Let Q be a set of n points arranged in a slightly perturbed unit square grid of size $\sqrt{n} \times \sqrt{n}$, so that the points are in general position. Define G_d as the geometric graph on Q where two points are connected if their distance is at most d. Then the number of crossings in G_d satisfies $\boxtimes(G_d) = \frac{\pi}{9} n d^6 (1 + o(1))$.*

Let Q be the set just described. First note that the k closest points to a point in Q not close to the boundary consist of those points inside a circle of radius $d = \sqrt{k/\pi} + \Theta(1)$. For the points close to the boundary, that is within d from it, their k closest points consist of those points in Q inside a circle of radius at most $2d$. Thus k-$\mathsf{NNG}(Q)$ has all the edges in G_d and some of the edges

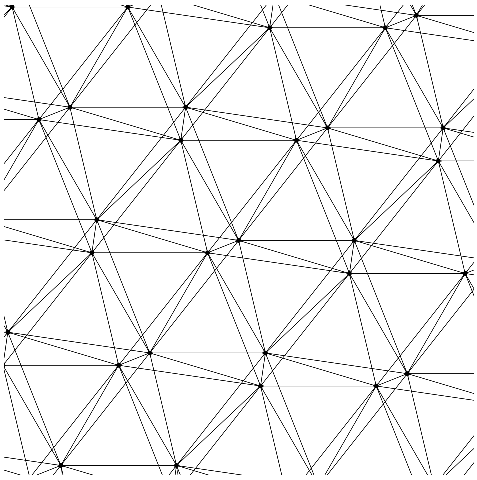

Figure 11: Point set whose 10-NNG has $4n + \Theta(\sqrt{n})$ crossings.

in G_{2d} whose endpoints are within d of the boundary. Thus

$$\begin{aligned}\boxtimes(k\text{-NNG}(Q)) &\leq \frac{\pi}{9}nd^6(1+o(1)) + \left(\frac{\pi}{9}n(2d)^6(1+o(1)) - \frac{\pi}{9}(\sqrt{n}-2d)^2(2d)^6(1+o(1))\right) \leq \\ &\leq \frac{\pi}{9}nd^6(1+o(1)) + \Theta(\sqrt{n}d^7) = \frac{\pi}{9}nd^6(1+o(1)) = \frac{1}{9\pi^2}k^3n(1+o(1)).\end{aligned}$$

Except for a similar analysis for the points close to the boundary, two points in Q are neighbors in k-RNG(Q) if their distance is at most $d = \sqrt{k/(2\pi/3 - \sqrt{3}/2)} + \Theta(1)$. Similarly, two points are neighbors in k-GG(Q) or in k-DG(Q) if their distance is at most $d = 2\sqrt{k/\pi}$. The result follows by the Proposition and by noting that the extra crossings caused by the points close to the boundary are at most $o(nd^6)$.

3.3.3 Proof of Theorem 2.3.3

Let e be an edge of k-DG(P). Let us see that there are many edges in k-DG(P) that do not cross e.

For the sake of simplicity, let us assume that e is horizontal. The line extending e divides P minus the endpoints of e into two groups. Let P_a and P_b respectively denote the set of points above and below the line. We set $|P_a| = l$. Observe that $|P_b| = n - l - 2$.

Let us first assume that $|P_a| \geq k+2$ and $|P_b| \geq k+2$. Let $p_1, p_2, \ldots, p_l$ denote the points in P_a sorted from top to bottom. If $i \in [2, k+2]$ and $j \in [1, i-1]$, then p_i is adjacent to p_j in k-DG(P). It suffices to consider the circle through p_i and p_j tangent to the horizontal line containing p_i. From all points in P, this circle can only contain $\{p_1, p_2, \ldots, p_{i-1}\} \setminus \{p_j\}$ in its interior. Notice that the edges $p_ip_1, p_ip_2, \ldots, p_ip_{i-1}$ do not cross e.

If $i \in [k+3, l]$, we consider the same family of circles. More precisely, we consider a circle tangent to the horizontal line through p_i growing until its interior contains $k+1$ points from P_a (it could happen that the interior of the circle goes from having k points from P_a to having $k+2$ points from P_a; this case is similar). Then in k-DG(P) these $k+1$ points are connected to p_i and all these edges do not cross e.

In conclusion, there exist $(k+1)(k+2)/2 + (l-(k+2))(k+1)$ edges between points in P_a not crossing e. By analogous arguments, there exist $(k+1)(k+2)/2 + (n-l-2-(k+2))(k+1)$ edges between points in P_b not crossing e. This adds up to a total number of $(k+1)(n-k-4)$ edges.

It remains to settle the case where either $|P_a| < k+2$ or $|P_b| < k+2$. Let us suppose that $|P_a| < k+2$; since $k < n/2 - 1$, we have that $|P_b| \geq k+1$. Arguing as in the previous case, we find $(l-1)l/2 + (k+1)(k+2)/2 + (n-l-2-(k+2))(k+1)$ edges that do not cross e. It is not difficult to see that, for any $l < k+2$, this number is always greater than $(k+1)(n-k-4)$.

In summary, since k-$\mathsf{DG}(P)$ contains at most $3(k+1)n - 3(k+1)(k+2)$ edges, e crosses no more than $3(k+1)n - 3(k+1)(k+2) - 1 - (k+1)(n-k-4)$ edges in k-$\mathsf{DG}(P)$. Hence the number of crossings of k-$\mathsf{DG}(P)$ is upper bounded by $\frac{1}{2}(3(k+1)n - 3(k+1)(k+2))\,(3(k+1)n - 3(k+1)(k+2) - 1 - (k+1)(n-k-4))$.

3.3.4 Proof of Proposition 2.3.4

Refer to Figure 12 (left). The upper chain contains $n-k-1$ points on a circle C such that the distance between consecutive points is constant. Let q_i, q_{i+1} be two such consecutive points. Let l be the line through q_{i+1} perpendicular to $\overrightarrow{q_i q_{i+1}}$, and let d be the distance between l and the center of C. The lower chain forms a convex chain seen from the upper chain and contains $k+1$ points that are at distance less than d from the center of C. This ensures that the closed disk with diameter given by q_i and some point from the lower chain does not contain any point from the upper chain different from q_i. Thus in k-$\mathsf{GG}(Q)$ each point from the upper group is adjacent to each point from the lower group. Notice that the construction can be perturbed to attain general position.

3.3.5 Proof of Proposition 2.3.5

Refer to Figure 12 (right). The number of points in the upper group is $k+1$, and the lower group contains the same number of points. The rest of points are placed in the middle group. In k-$\mathsf{DG}(Q)$ each point q_i in the middle group is connected to all upper and lower points, as it suffices to consider families of increasing circles through q_i with center at the vertical line through q_i. This construction can be perturbed so that it becomes non-degenerate.

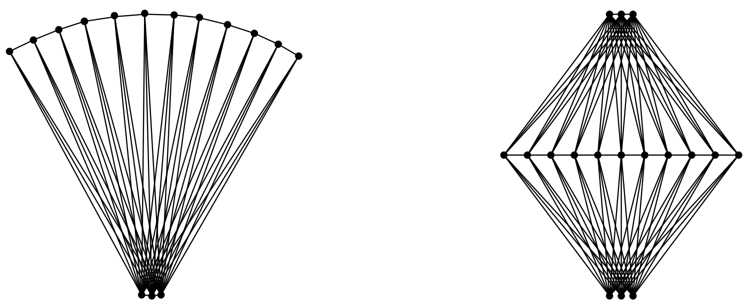

Figure 12: Left: set of points whose k-GG has $k^2n^2/4 + o(k^2n^2)$ crossings. Right: set of points whose k-DG has $k^2n^2/2 + o(k^2n^2)$ crossings.

3.3.6 Proof of Theorem 2.3.6

Lemma 3.3.3. *In any angular sector with apex $p \in P$ and amplitude $\alpha \leq \pi/3$, the only points that can be connected to p in the graph k-$\mathsf{RNG}(P)$ are the $k+1$ closest points to p that are contained in the sector.*

Proof. Let $p_1, p_2, \ldots$ be the points of P that are contained in the sector sorted by increasing distance to p. For each $i \geq 2$, the points $p_1, p_2, \ldots, p_{i-1}$ are contained in the intersection of the two disks centered at p, p_i with radius $|pp_i|$ (see Figure 13, left). Consequently, p and p_i are not connected in k-$\mathsf{RNG}(P)$ if $i - 1 > k$. □

Let $e = p_i p_j$ be an edge in k-$\mathsf{RNG}(P)$. We define the *lens* associated to e as the open intersection of the circles centered at p_i and p_j with radius $|p_i p_j|$*.

Now let us define a charging scheme that assigns every crossing in k-$\mathsf{RNG}(P)$ to each of the two involved edges e satisfying that at least one of the endpoints of the other edge is contained in the lens associated to e. Since each crossing defines a quadrilateral having at least one obtuse angle, the crossing is (at least) assigned to the edge opposite to this angle.

Let e be an edge in k-$\mathsf{RNG}(P)$. The lens associated to e contains at most k points in P. By Lemma 3.3.3, each of them is adjacent to no more than $3k + 3$ points in P such that the edge that connects them crosses e. Consequently, at most $3k^2 + 3k$ crossings may be assigned to e.

Since each vertex in P has degree at most $6k + 6$ (see Lemma 3.3.3), the number of edges of k-$\mathsf{RNG}(P)$ does not exceed $3kn + 3n$, which yields the theorem.

3.3.7 Proof of Theorem 2.3.7

The proof of Theorem 2.3.7 requires several technical lemmas. The first one is a corollary of Lemma 3.3.3:

Lemma 3.3.4. *In any angular sector with apex $p \in P$ and amplitude $\alpha \leq \pi/3$, the only points that can be connected to p in the graph k-$\mathsf{NNG}(P)$ are the k closest points to p that are contained in the sector.*

Lemma 3.3.5. *Let p_i, p_j, p_l, p_m be four elements of P, with $|p_i p_j| < |p_i p_l| < |p_i p_m|$. If $p_j p_m$ crosses $p_i p_l$, then $|p_j p_l| < |p_j p_m|$.*

Proof. Let $C_{i,l}$ and $C_{j,l}$ respectively be the circles centered at p_i and p_j containing p_l in the boundary (see Figure 13, middle). These circles have a non empty intersection and, since p_i, p_j, and p_l are not aligned, $C_{i,l}$ is not contained in $C_{j,l}$, nor $C_{j,l}$ is contained in $C_{i,l}$. Let q be the intersection point between $C_{i,l}$ and the ray starting at p_j and passing through p_i. We have that $|p_j q| > |p_j p_l|$. Therefore the ray starting at p_j and passing through p_i intersects $C_{j,l}$ before $C_{i,l}$. This property is maintained for all rays starting at p_j and contained in the wedge induced by the angle $\angle p_i p_j p_l$. In particular, it is maintained for the ray starting at p_j and containing p_m. Since p_m lies outside $C_{i,l}$, p_m is not contained in $C_{j,l}$, so $|p_j p_l| < |p_j p_m|$. □

Lemma 3.3.6. *Let p_i, p_j, p_l be three elements of P, with $|p_i p_j| < |p_i p_l|$. Then all points p_m such that $p_j p_m$ crosses $p_i p_l$, $|p_i p_l| < |p_i p_m|$, $|p_m p_i| > |p_m p_j|$, and $|p_m p_l| > |p_m p_j|$ are contained in an angular sector with apex p_i and amplitude at most $\pi/3$.*

Proof. Without loss of generality, we assume that the line through p_i and p_l is vertical, p_i is above p_l, and p_j is to the left of this line. The other situations are symmetric.

Let C be the circle centered at p_i and containing p_l in the boundary. Let $l_{i,j}$ and $l_{j,l}$ respectively be the bisectors of $p_i p_j$ and $p_j p_l$. A point p_m satisfying the hypothesis of the lemma lie on the intersection R of the following four regions: the exterior of C, the semiplane opposite to p_i

*Unfortunately, it is standard in the computational geometry literature that a *lens* is incorrectly called a *lune*.

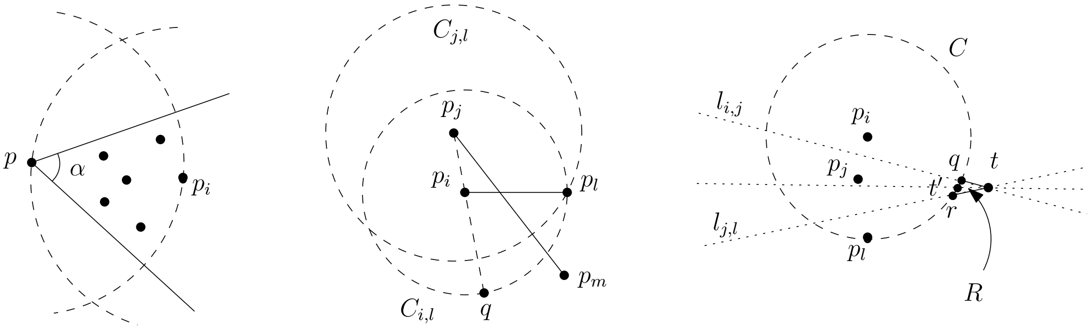

Figure 13: Left: an angular sector with apex p and amplitude $\alpha \leq \pi/3$. Middle: four points satisfying the hypothesis of Lemma 3.3.5. Right: the region R in Lemma 3.3.6.

determined by $l_{i,j}$, the semiplane opposite to p_l determined by $l_{j,l}$, and the wedge induced by the angle $\angle p_i p_j p_l$ (see Figure 13, right).

If p_j has greater or equal ordinate than p_i, it is not difficult to see that region R is empty. Observe that R is also empty if $l_{i,j}$ or $l_{j,l}$ do not intersect the arc of C determined by the wedge induced by $\angle p_i p_j p_l$. Therefore the lemma clearly holds in these cases.

Let us now suppose that p_j has smaller ordinate than p_i, $l_{i,j}$ intersects the arc of C determined by the wedge induced by $\angle p_i p_j p_l$ in a point q, and $l_{j,l}$ intersects the arc of C determined by the wedge induced by $\angle p_i p_j p_l$ in a point r. Let t be the intersection of $l_{i,j}$ and $l_{j,l}$. In order for R not to be empty t must lie outside C. Let us assume that we are in this situation.

Consider the wedge formed by the ray starting at p_j and passing through q together with the ray starting from p_j and passing through r. Observe that R is contained in this wedge. We will end the proof by showing that this wedge has angle at most $\pi/3$. Let t' be the intersection of the bisector of $p_i p_l$ with the the arc of C determined by the wedge induced by $\angle p_i p_j p_l$. Notice that $\angle p_j p_i q > \angle p_j p_i t' > \angle p_l p_i t'$. By analogous arguments, $\angle p_j p_l r > \angle p_i p_l t'$. Since $\angle p_l p_i t'$ and $\angle p_i p_l t'$ are angles of the equilateral triangle formed by p_i, p_l, and t', then $\angle p_j p_i q$ and $\angle p_j p_l r$ are greater than $\pi/3$. This implies that $\angle p_i p_j q$ and $\angle p_l p_j r$ are greater than $\pi/3$, since $\angle p_j p_i q = \angle p_i p_j q$ and $\angle p_l p_j r = \angle p_j p_l r$. Given that $\angle p_i p_j q + \angle q p_j r + \angle r p_j p_l < \pi$, we conclude that $\angle q p_j r < \pi/3$. $\square$

Now we are ready to prove Theorem 2.3.7.

Consider two crossing edges in k-$\mathsf{NNG}(P)$ involving vertices p_i, p_j, p_l, p_m. We assign the crossing to each of the pairs of edges $\{\overrightarrow{p_i p_l}, \overrightarrow{p_i p_j}\}$ satisfying: (i) one of the two crossing edges is $\overrightarrow{p_i p_l}$; (ii) $|p_i p_j| < |p_i p_l|$ (so $\overrightarrow{p_i p_j} \in E(k\text{-}\mathsf{NNG}(P))$).

Let us show that this assignment is consistent. The quadrilateral defined by the vertices involved in the crossing has at least one obtuse angle. Then the crossing is assigned to the pair of directed edges consisting of the edge opposite to this obtuse angle (which is a diagonal of the quadrilateral) and one edge with the same origin and lying in one side of the quadrilateral.

We devise a charging scheme that divides the weight of each crossing by the number of pairs of edges the crossing is assigned to. We say that a crossing is *simple* if it is only assigned to one pair of edges, and we say that it is *multiple* otherwise. In the following we find the maximum weight that a pair of edges can receive.

Let p_j and p_l be two of the k nearest neighbors of p_i, with $|p_i p_j| < |p_i p_l|$. Each crossing assigned to $\{\overrightarrow{p_i p_l}, \overrightarrow{p_i p_j}\}$ can be associated with a vertex adjacent to p_j in k-$\mathsf{NNG}(P)$ (the fourth point involved in the crossing). We want to bound the maximum number of such vertices. Let $\hat{w}$ be the wedge induced by $\angle p_i p_j p_l$. If $\hat{w}$ has amplitude at most $2\pi/3$, then, by Lemma 3.3.4, the maximum number of crossings that may be assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$ is $2k$. Otherwise we partition $\hat{w}$ into three wedges $\hat{w}_1$, $\hat{w}_2$, and $\hat{w}_3$ as follows: $\hat{w}_1$ is bounded by the half-line with origin at p_j and direction given by $\overrightarrow{p_jp_i}$ and has amplitude $pi/3$; $\hat{w}_3$ is bounded by the half-line with origin at p_j and direction given by $\overrightarrow{p_jp_l}$ and has amplitude $pi/3$; $\hat{w}_2$ consists of the part of $\hat{w}$ not covered by $\hat{w}_1$ and $\hat{w}_3$. For $\nu \in \{1,2,3\}$, let n_ν be the number of vertices in $\hat{w}_\nu$ that create a crossing assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$. A direct application of Lemma 3.3.4 yields that $n_\nu \leq k$ for $\nu \in \{1,2,3\}$. Furthermore, since p_i is a point in $\hat{w}_1$ adjacent to p_j, we have that $n_1 \leq k-1$. Finally, consider the k closest points to p_j contained in $\hat{w}_3$, which, by Lemma 3.3.4, are the only candidates to be connected to p_j in k-$\mathsf{NNG}(P)$. Observe that p_l belongs to this set: otherwise, by Lemma 3.3.5, there would be k points p_m such that $|p_ip_l| > |p_ip_m|$, which is absurd because $\overrightarrow{p_ip_l} \in E(k\text{-}\mathsf{NNG}(P))$. Thus $n_3 \leq k-1$. In conclusion, the maximum number of crossings that may be assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$ is $3k-2$.

Next we analyze the maximum number of simple crossings that may be assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$. Let p_m be a vertex in P such that the edge p_jp_m (with some orientation) causes a crossing assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$. If $|p_ip_m| < |p_ip_l|$, then the crossing is also assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_m}\}$. If $|p_ip_m| > |p_ip_l|$ and $\overrightarrow{p_jp_m} \in E(k\text{-}\mathsf{NNG}(P))$, then, by Lemma 3.3.5, $|p_jp_l| < |p_jp_m|$ and the crossing is also assigned to $\{\overrightarrow{p_jp_m}, \overrightarrow{p_jp_l}\}$. If $|p_ip_m| > |p_ip_l|$, $\overrightarrow{p_mp_j} \in E(k\text{-}\mathsf{NNG}(P))$, and $|p_mp_j| > |p_mp_i|$ or $|p_mp_j| > |p_mp_l|$, then the crossing is also assigned to $\{\overrightarrow{p_mp_j}, \overrightarrow{p_mp_i}\}$ or $\{\overrightarrow{p_mp_j}, \overrightarrow{p_mp_l}\}$. Therefore three necessary conditions for p_m to cause a simple crossing assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$ are: (i) $|p_ip_m| > |p_ip_l|$; (ii) $\overrightarrow{p_mp_j} \in E(k\text{-}\mathsf{NNG}(P))$; (iii) $|p_mp_i| > |p_mp_j|$ and $|p_mp_l| > |p_mp_j|$. By Lemmas 3.3.6 and 3.3.4, there are at most k such points.

To conclude, in the worst case k simple crossings and $2k-2$ crossings of weight $1/2$ are assigned to $\{\overrightarrow{p_ip_l}, \overrightarrow{p_ip_j}\}$. Thus any pair of edges in k-$\mathsf{NNG}(P)$ receives weight at most $2k-1$.

3.3.8 Proof of Proposition 2.3.8

Consider the set $Q = \{q_1, q_2, \ldots, q_n\}$, where $q_i = (2^i, 0)$. Let us slightly perturb the configuration so that the points are in convex position. The k nearest neighbors of each point q_i such that $i > k$ are $q_{i-1}, q_{i-2}, \ldots, q_{i-k}$. Therefore, if $i \in [k+1, n-k]$, then q_i is connected in k-$\mathsf{NNG}(Q)$ to its k predecessors and k successors in the "line".

Let $\overrightarrow{q_iq_j}$, $\overrightarrow{q_lq_m}$ be two crossing edges such that $j < m < i < l$. We assign this crossing to q_j. Suppose that $j \in [k+1, n-k]$. Then the crossings between $\overrightarrow{q_iq_j}$ and the following edges are assigned to q_j: $\overrightarrow{q_{j+1}q_{j-1}}, \overrightarrow{q_{j+2}q_{j-1}}, \ldots, \overrightarrow{q_{j+k-1}q_{j-1}}$, $\overrightarrow{q_{j+1}q_{j-2}}, \overrightarrow{q_{j+2}q_{j-2}}, \ldots, \overrightarrow{q_{j+k-2}q_{j-2}}, \ldots,$ $\overrightarrow{q_{j+1}q_{i+1}}, \overrightarrow{q_{j+2}q_{i+1}}, \ldots,$ $\overrightarrow{q_{i+k+1}q_{i+1}}$. This adds up to $\sum_{\nu=i+k+1-j}^{k-1} \nu$ crossings. Since i might take values from $j-k$ to $j-2$, the total number of crossings assigned to q_j is

$$\sum_{i=j-k}^{j-2} \sum_{\nu=i+k+1-j}^{k-1} \nu = \sum_{\nu=1}^{k-1} \nu^2 = \frac{k^3}{3} - \frac{k^2}{2} + \frac{k}{6}.$$

Given that j is an index in $[k+1, n-k]$, the preceding charging scheme guarantees that k-$\mathsf{NNG}(Q)$ contains $(n-2k)\left(k^3/3 - k^2/2 + k/6\right)$ crossings. Notice that the crossings we have not account for in this argument have order $o(k^3 n)$.

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